

Stochastic Volatility for Lévy Processes

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OUTLINE

- Objectives for Option Price Modeling.
 - Explaining as simply as possible the variation in option prices across time, strike and maturity.
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 - The opposite extremes.
- The FVIMA starting point.
 - The relevance of changing time.
 - The importance of the absence of a diffusion component, theoretically and empirically.
 - The issue of dimensionality and filtration.

- The difficulty of calibrating Homogeneous Lévy Processes across maturities.
 - Theoretical behavior across maturity.
 - Summary of some empirical observations across maturity.
 - The resulting inconsistencies and modeling necessities.

- The primary considerations of variation across time.
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 - Modeling the clustering of uncertainty using clustered time.
 - The CIR process as the engine for this property.

- Summary of The Lévy processes considered.
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- Stochastic volatility and the processes $NIGSV$, $VGSV$, $CGMYSV$.

- Exponentiating the SV processes to form stock price return models over different horizons.
 - Ordinary Exponentials $NIGSA$, $VGSA$, $CGMYSA$.
 - Stochastic Exponentials $NIGSAM$, $VGSAM$, $CGMYSAM$.

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Objectives for Option Price Modeling

- Focusing on call options we may write the prices as functions of time, t , strike K , and maturity T ,

$$C(t, K, T)$$

- We would like a model that works for all strikes, maturities and does so consistently through calendar time.
 - Option Prices have the most structure across the strike domain, where they have to be convex in the strike variable.
 - The next structurally important direction is maturity where there are restrictions associated with positive forward variance.

- We find the least structure in calendar time as this variable in fact summarizes variations in the underlying risks.
- Put alternatively, it is more than variations in the current spot price that will be needed to explain the evolution of the surface of option prices and we first need to discover the additional variables whose time evolution is critical to understanding movements in option prices.
- Our strategy is to
 - * build on success in the strike direction,
 - * followed by improving performance in the maturity direction,
 - * leaving for now the calendar time direction for future research.

- Black-Merton-Scholes simplicity.
 - For equity index options we have some 200 options covering 20 strikes across 10 maturities.
 - The Black-Merton-Scholes theory implies that the dimensional content of this information is *one* represented by any one of the implied volatilities.
 - Such a dimensional reduction is in our view an extremely unrealistic expectation that is now confirmed by numerous studies and the very existence of the volatility surface documents this fact.

- The other extreme.
 - A prime example of the other extreme is the local volatility model that places option prices across the strike maturity spectrum in 1 – 1 correspondence with the local volatility of the stock at future dates for varying levels of the underlying in an assumed diffusive development of the stock price dynamics.
 - Attempts are being made to reduce dimensions here by fixing in rigid parametric classes the local volatility function.
 - We question the assumed relevance of a diffusive development from a variety of perspectives.

The FVIMA starting point.

- We recognize that price processes in the absence of arbitrage are semimartingales.
- Such processes are time changed Brownian motion and it is argued in Geman, Madan and Yor (2001) that if the time change has a martingale component then it is purely discontinuous.

- Many scholars have concluded that one must have a jump component and have gone on to build the “so-called” jump-diffusion models and we cite the recent interesting developments made by Bates (2000) and Duffie, Pan and Singleton (2000).
- Jump diffusion approaches model the large number of small movements using a diffusion process and combine this process with an orthogonal jump process to model the possibility of relatively infrequent large moves.
- Given the existence of jump processes with infinitely many small moves as well as finitely many large moves we question the relevance of adding a diffusion component in the presence of a high activity jump process that unifies both types of movements.

- Carr, Geman, Madan and Yor (2001) test for the existence of a diffusion component in the presence of an infinite activity Lévy process and find that
 - this component may be dispensed with in the presence of an infinite activity Lévy density.
 - the price process is by and large one of finite variation with a completely monotone Lévy density.
 - the resulting homogeneous Lévy processes adequately calibrate across a wide range of strikes.

- We anticipate that the apparent dimensional content of option prices on say the *S&P* 500 index that is approximately 200 can be reduced to the variation across time of 6 to 10 parameters that successfully calibrates the surface each day.
 - The dynamics of option prices across calendar time either statistically or risk neutrally is then to be explained in terms of the evolution over time of these parameters.
 - The principle for modeling the daily surface may well be just the absence of static arbitrage opportunities leaving the absence of dynamic arbitrage to a later stage of research that first identifies the relevant filtration within which to discuss the question of the absence of dynamic arbitrage opportunities.
 - Put alternatively, option prices are observed by market participants and they may well be part of the filtration describing the evolution of the stock prices and option prices in a simultaneous description of this dynamics.

The impossibility of calibrating Homogeneous Lévy Processes across maturities.

- The log characteristic function of homogeneous Lévy processes is linear in time to maturity.
- This property has the easily computed consequence that
 - i*) the t period annualized volatility of log returns is constant,
 - ii*) the absolute skewness of t period log returns is proportional to $t^{-0.5}$,
 - iii*) excess kurtosis or kurtosis-3 is proportional to t^{-1} .

- The following figures show the quarterly average moments (annualized volatility, absolute skewness and excess kurtosis) for the risk neutral density as functions of time to expiration for S&P500 Index in 1999.

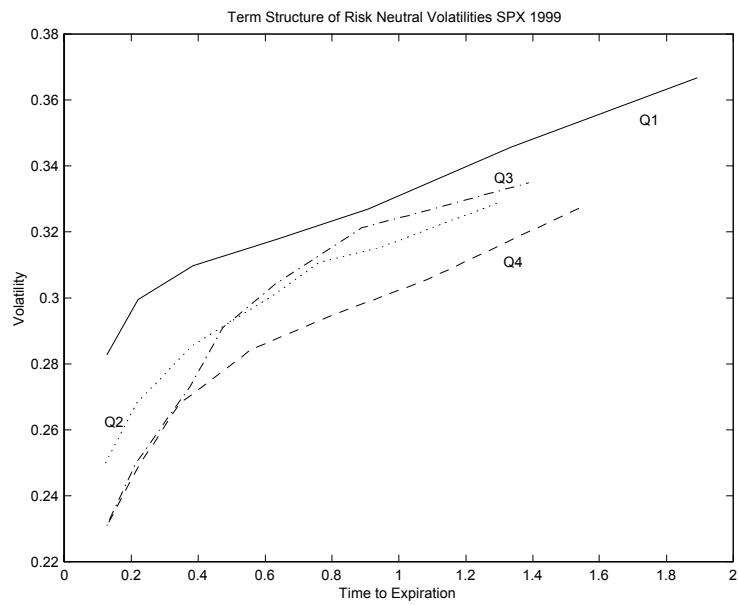


Figure 1: Term Structure of Volatility.

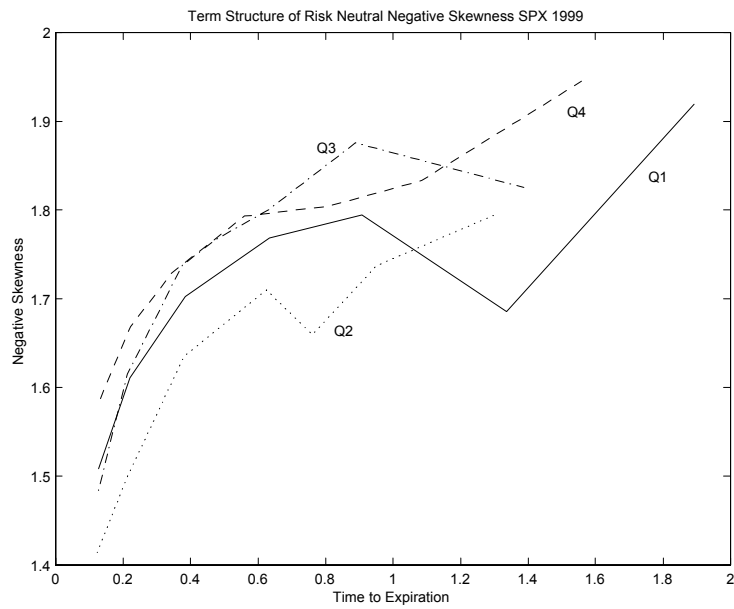


Figure 2: Term Structure of Absolute Skewness.

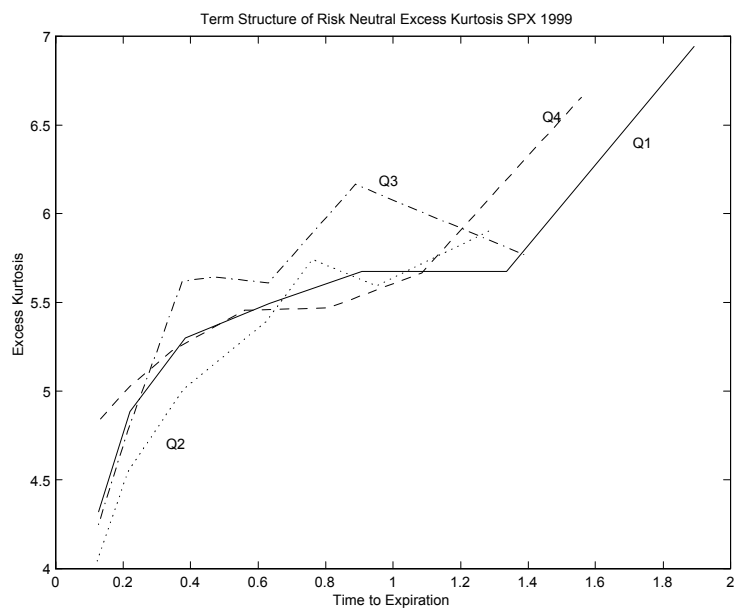


Figure 3: Term Structure of Excess Kurtosis.

- We can easily see from these graphs that the respective moments are increasing in time to maturity.
- These observations are inconsistent with the assumption that log returns follow a homogeneous Lévy process.

The primary considerations of variation across time.

- In the Black Merton Scholes model price uncertainty is modeled by the local volatility of the stock return.
- Many authors including Engle (1982), Heston (1993), Bates (1996), Barndorff-Nielsen and Shepard (1999) have examined and concluded that the local characteristic of uncertainty is itself uncertain over time and displays a tendency to cluster.
- For a Lévy process the local uncertainty is richer, being multidimensional and displaying volatility, skewness, and kurtosis.

- We aggregate the dimensions of uncertainty of the Lévy process into time itself.
- We increase or decrease the level of uncertainty by speeding up or slowing down the rate at which time passes.
- To build clustering and to keep time going forward we employ a mean reverting positive process as a measure of the local rate of time change.
- The specific process we use is the “so-called” CIR process.

Summary of The Lévy processes considered.

1. The Normal Inverse Gaussian Model (Barndorff-Nielsen, (1998)).

- Define by T_t^ν the time it takes Brownian motion with drift ν to reach the level t . It is well known that

$$\begin{aligned} & E [\exp(-\lambda T_t^\nu)] \\ &= \exp\left(-t \left((2\lambda + \nu^2)^{1/2} - \nu\right)\right) \end{aligned}$$

- Now evaluate another independent Brownian motion with drift θ and volatility σ at T_t^ν to get the *NIG* process

$$X_{NIG}(t; \sigma, \nu, \theta) = \theta T_t^\nu + \sigma W(T_t^\nu)$$

- The characteristic function is

$$\begin{aligned}
 \phi_{NIG}(u; \alpha, \beta, t\delta) &= \exp(-t\delta(A - B)) \\
 A^2 &= \alpha^2 - (\beta - iu)^2 \\
 B^2 &= \alpha^2 - \beta^2 \\
 \beta &= \frac{\theta}{\sigma^2} \\
 \alpha^2 &= \frac{\nu^2}{\sigma^2} + \frac{\theta^2}{\sigma^4} \\
 \delta &= \sigma
 \end{aligned}$$

- The Lévy density is

$$k_{NIG}(x) = \left(\frac{2}{\pi}\right)^{1/2} \delta \alpha^2 \frac{e^{\beta x} K_1(x)}{|x|}$$

- The unit time log characteristic function is

$$\psi_{NIG}(u; \sigma, \nu, \theta) = \sigma \left(\frac{\nu}{\theta} - \left(\frac{\nu^2}{\theta^2} - 2\frac{\theta i u}{\sigma^2} + u^2 \right)^{1/2} \right)$$

2. The Variance Gamma Model (Madan, Carr and Chang, (1998)).

- Define by G_t^ν the gamma process with mean rate unity and variance rate ν . It is well known that

$$E [\exp (-\lambda G_t^\nu)] = (1 + \lambda\nu)^{-t/\nu}$$

- Now evaluate another independent Brownian motion with drift θ and volatility σ at T_t^ν to get the VG process

$$X_{NIG}(t; \sigma, \nu, \theta) = \theta G_t^\nu + \sigma W(G_t^\nu)$$

- The characteristic function is

$$\begin{aligned} & \phi_{VG}(u; \sigma, \nu, \theta) \\ &= \left(1 - iu\theta\nu + \sigma^2\nu u^2/2\right)^{-t/\nu} \end{aligned}$$

- The Lévy density is

$$k_{VG}(x) = \begin{cases} \frac{C \exp(Gx)}{|x|} & x < 0 \\ \frac{C \exp(-Mx)}{|x|} & x > 0 \end{cases}$$

$$C = \frac{1}{\nu}$$

$$G = \left(\left(\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2} \right)^{1/2} - \frac{\theta\nu}{2} \right)^{-1}$$

$$M = \left(\left(\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2} \right)^{1/2} + \frac{\theta\nu}{2} \right)^{-1}$$

- The unit time log characteristic function is

$$\begin{aligned} & \psi_{VG}(u; C, G, M) \\ &= C \log \left(\frac{GM}{GM + (M - G)iu + u^2} \right) \end{aligned}$$

3. The CGMY process (Carr, Geman, Madan and Yor, forthcoming).

- This process is defined by generalizing the VG Lévy density to

$$k_{CGMY}(x) = \begin{cases} \frac{C_n \exp(Gx)}{|x|^{1+Y_n}} & x < 0 \\ \frac{C_p \exp(Mx)}{|x|^{1+Y_p}} & x > 0 \end{cases}$$

- The characteristic function is

$$\begin{aligned} & \phi_{CGMY}(u; C_n, G, Y_n, C_p, M, Y_p) \\ &= \exp \left(\begin{array}{l} tC_p \Gamma(-Y_p) ((M - iu)^{Y_p} - M^{Y_p}) \\ + tC_n \Gamma(-Y_n) ((G + iu)^{Y_n} - M^{Y_n}) \end{array} \right) \end{aligned}$$

- The unit time log characteristic follows easily.

Stochastic volatility and the processes NIGSV, VGSV, CGMYSV.

- We model the rate of time change by the process $y(t)$ that solves the stochastic differential equation

$$dy = \kappa(\eta - y)dt + \lambda y^{1/2}dW$$

- The economic time elapsed in t units of calendar time is then given by $Y(t)$ where

$$Y(t) = \int_0^t y(s)ds$$

- The characteristic function of $Y(t)$ is known as

$$\begin{aligned}
 & \phi_Y(u, t, y(0); \kappa, \eta, \lambda) \\
 &= A(t, u) \exp(b(t, u)y(0)) \\
 A(t, u) &= \exp\left(\frac{\kappa^2 \eta t}{\lambda^2}\right) \\
 & \quad \times \left(\cosh\left(\frac{\gamma t}{2}\right) + \frac{\kappa}{\gamma} \sinh\left(\frac{\gamma t}{2}\right) \right)^{-\frac{2\kappa\eta}{\lambda^2}} \\
 b(t, u) &= \frac{2iu}{\kappa + \gamma \coth\left(\frac{\gamma t}{2}\right)}
 \end{aligned}$$

- For $X(t)$ a Lévy process, we define the stochastic volatility Lévy process (*SVLP*), by

$$Z(t) = X(Y(t))$$

- Let the unit time log characteristic function for the Lévy process be $\psi_X(u)$, we have that the characteristic function for Z is given by

$$\begin{aligned}\phi_Z(u) &= E[\exp(iuX(Y(t)))] \\ &= E[\exp(Y(t)\psi_X(u))] \\ &= \phi_Y(-i\psi_X(u), t, y(0); \kappa, \eta, \lambda)\end{aligned}$$

- Taking for X the processes NIG , VG , and $CGMY$ we obtain the the processes $NIGSV$, $VGSV$, and $CGMYSV$. For NIG the parameter σ absorbs $y(0)$, while for VG and $CGMY$ the parameter C absorbs $y(0)$. The final parameter sets are

$$\begin{array}{ll}
 NIGSV & \sigma, \nu, \theta, \kappa, \eta, \lambda \\
 VGSV & C, G, M, \kappa, \eta, \lambda \\
 CGMYSV & C, G, M, Y_p, Y_n, \zeta, \kappa, \eta, \lambda
 \end{array}$$

- We should entertain an expansion of the size of the parameter space only if the improvement in the empirical performance of the model is substantial to justify the costs of interpreting and managing the evolution of a higher dimensional information filtration.

Exponentiating the SV processes to form stock price return models over different horizons.

1. The Ordinary Exponential

- We model the risk neutral stock price as an exponential of the process $Z(t)$, enforcing spot forward arbitrage restrictions by normalization as follows

$$\begin{aligned} S(t) &= S(0) \frac{\exp((r - q)t + Z(t))}{E[\exp(Z(t))]} \\ &= S(0) \exp((r - q)t) N(t) \\ E[\exp(Z(t))] &= \phi_Y(-i\psi_X(-i), t, y(0); \kappa, \eta, \lambda) \end{aligned}$$

- In this approach we are pricing at time 0 claims written on the paths of $(S(t), t \geq 0)$ by evaluating a discounted expectation of the payoffs under a specific measure on the space of paths described at time 0.

- Even more specifically, we do not say anything about the price of the claim the next day and we do not require discounted prices to be martingales with respect to some filtration. In fact we do not here even describe this filtration.
- With respect to arbitrage considerations we just impose the absence of static arbitrages and calendar spread arbitrages that may be triggered by strategies described completely at time 0 in terms of the stock price at pre-specified future dates.
- Note that $N(t)$ is a process of unit expectation but it is generally not a martingale.
- The special cases for our three Lévy processes we term the *NIGSA*, *VGSA*, *CGMYSA*.

2. The Stochastic Exponential

- Here we consider the joint filtration of the time change $Y(t)$ and the stock price and impose the absence of dynamic arbitrage in this filtration by adopting a martingale model.

- Define a jump compensator by the process

$$\rho(dx, dt) = y(t)k(x)dxdt$$

- Consider the compensated jump martingale

$$m(t) = (e^x - 1) * (\mu - \rho)$$

where μ is the integer valued random measure associated with the jumps of a process with compensator ρ .

- The stochastic exponential of $m(t)$ is given by the positive martingale $M(t)$ where

$$\begin{aligned}
 & M(t) \\
 &= \exp\left(Z(t) - \int_0^t \int_{-\infty}^{\infty} (e^x - 1)y(t)k(x)dxdt\right) \\
 &= \exp(Z(t) - Y(t)\psi_X(-i))
 \end{aligned}$$

- Equivalently we evaluate the martingale

$$\exp(X(t) - t\psi_X(-i))$$

at $Y(t)$ to obtain $M(t)$.

- In this approach we model the stock price by

$$S(t) = S(0) \exp((r - q)t) M(t)$$

- The process $M(t)$, unlike $N(t)$ is here a martingale in the joint filtration of $(S(t), Y(t))$.
 - If parameters are constant over time and the only risks embedded in the surface of option prices are the movements in the price $S(t)$ and variations in the level of economic time $Y(t)$ then we have here a model for option prices through time that is free of arbitrage opportunities.
 - To the extent $Y(t)$ is unobservable or other parameters change, the relevance of the martingale model is called into question.
- The special cases for our three Lévy processes we term *NIGSAM*, *VGSAM*, *CGMYSAM*.

The SA Martingale Laws.

- The absence of arbitrage is associated with martingale laws.
- Martingale laws are defined with respect to certain filtrations.
- Option prices really only specify the one dimensional distributions of the risk neutral stock price at various time points given by the option maturities.
- We ask here when we may tear up the original filtration of a process and construct a martingale in another filtration but without changing the one dimensional marginal distributions and thus keeping the same option prices.

- In our context the constant expectation process $N(t)$ is derived as

$$N(t) = \frac{\exp(X(Y(t)))}{E[\exp(X(Y(t)))]}$$

- If we can construct a process $U(t)$ of independent and homogeneous increments that has the same one dimensional distributions as the process $Y(t)$ then

$$\tilde{N}(t) = \frac{\exp(X(U(t)))}{E[\exp(X(U(t)))]}$$

is a martingale with the same one dimensional laws as $N(t)$.

- We focus attention on the one-dimensional laws of $Y(t)$.

- This leads us to the definition of the Lévy marginal (LM) property whereby we say that

$$H(t) = \int_0^t h(u) du$$

has the LM property if there exists an inhomogeneous Lévy process $\theta(t)$ such that

$$H(t) \stackrel{(d)}{=} \theta(t)$$

- We show that the process $Y(t)$ has the LM property when $y(0) = 0$.
- However this is not the case when $y(0) = x \neq 0$.

- We note that when $X(t)$ is a homogeneous Lévy process then provided one finds a process of independent and possibly inhomogeneous increments $\theta(t)$ such that

$$Y(t) \stackrel{(d)}{=} \theta(t)$$

then

$$X(Y(t)) \stackrel{(d)}{=} X(\theta(t)),$$

but the latter is a process of independent increments and so

$$N(t) = \frac{\exp(X(\theta(t)))}{E[\exp(X(\theta(t)))]}$$

is both a process of unit expectation and a martingale.

- The analysis then turns on obtaining the property

$$Y(t) \stackrel{(d)}{=} \theta(t)$$

for some process $\theta(t)$.

- A special case is particularly instructive. Consider the process

$$Y_0(t) = \int_0^t W(s)^2 ds$$

- The Laplace transform of $Y_0(t)$ in $\lambda^2/2$ is

$$E \left[\exp \left(-\frac{\lambda^2}{2} Y_0(t) \right) \right] = \cosh(\lambda t)^{-1}.$$

- This function is however also the Laplace transform in $\lambda^2/2$ of the first hitting time of reflecting Brownian motion to the level t or the process

$$\theta(t) = \inf \{s \mid |\beta(s)| > t\}$$

for an independent Brownian motion $\beta(t)$.

- The process $\theta(t)$ is an inhomogeneous pure jump Lévy process and so for example

$$X_1(t) = W(Y_0(t))$$

$$X_2(t) = W(\theta(t))$$

are examples of continuous and purely discontinuous processes that yield the same option prices for all strikes and maturities but have different path properties as one is continuous and the other is purely discontinuous.

- More generally we consider the property of martingale marginals (MM) and we say that a process of constant expectation $H(t)$ has the MM property just if there exists a martingale with the same one-dimensional distributions.
- We conjecture that $NIGSA$, $VGSA$, and $CGMYSA$ have the MM property and this is currently under investigation.

Data Summary and Sample of Results.

- For data on option prices across all strikes and maturities up to one year on 20 underlying assets for every second wednesday of every month for the year 2000 we estimate the six models

NIGSA, VGSA, CGMYSA

NIGSAM, VGSAM, CGMYSAM

- We use the (Carr and Madan, (1998)) method of pricing options by inverting, using the *FFT* (Fast Fourier Transform), the Fourier transform of the modified call price seen as a function of log strike and defined by

$$\gamma(u, t) = \int_{-\infty}^{\infty} e^{iuk} e^{\alpha k} C(k, t) dk.$$

The inverse transform is applied uniformly across all maturities and yields the surface in a single pass of the *FFT*.

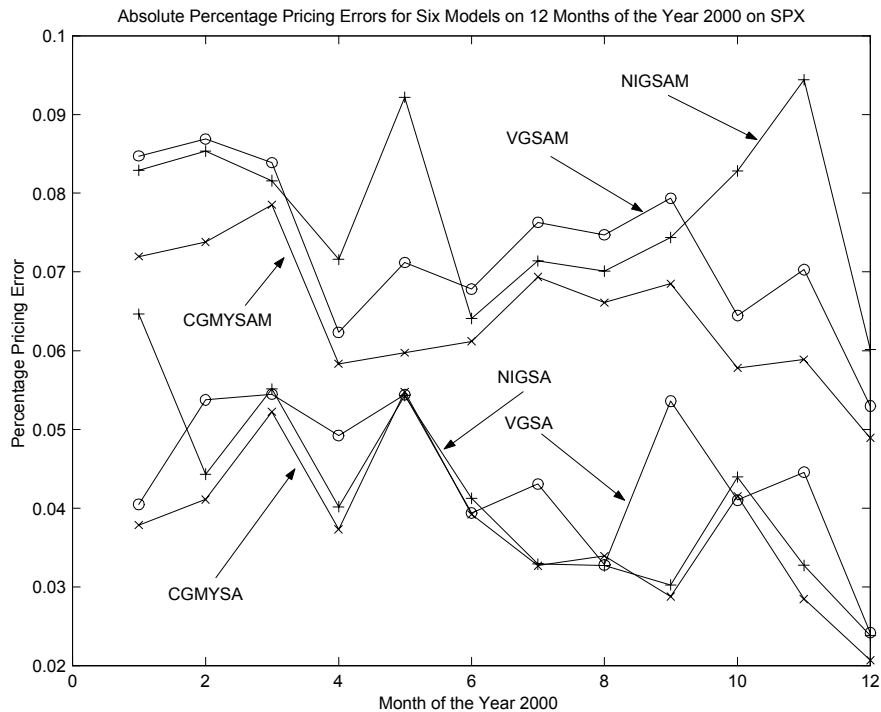


Figure 4: Graphs of the absolute percentage errors for the six models on the SPX for the year 2000

- We present here a summary graph of the absolute percentage errors for the six models on the *SPX* for the year 2000.
- The clear domination of the exponential over the stochastic exponential is apparent.

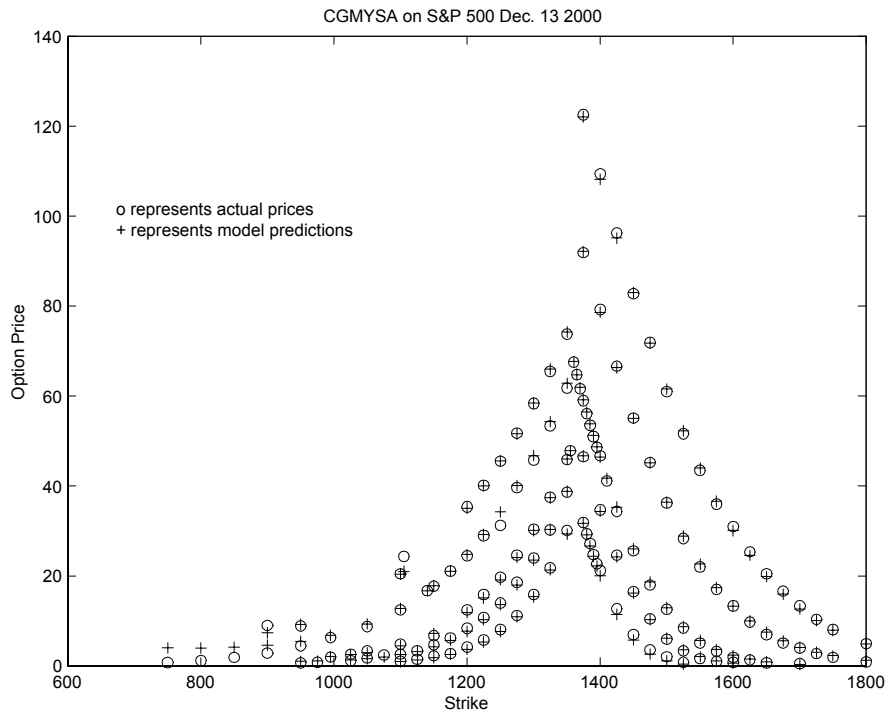


Figure 5: Graph of Price fit on OTM options on SPX for December 13 2000.

- We also present a graph of the price fit for *CGMYSA* on December 13 2000.

Discussion of Modeling Strategy

- There is yet too much to learn about option prices at a point of time to successfully venture into modeling the dynamic filtration with respect to which these prices may be martingales.
- We should obtain parsimonious models that satisfactorily capture the surface each day and then study the time evolution of the parameters.
- The movements in the parameters of the model capturing the daily surface is the relevant filtration for the dynamic behavior of option prices. In this sense we view option prices as part of the filtration as they lead to assessments in the variations of the underlying parameters.

- When capturing the daily surface the restriction to martingale models in some presumed filtration is unduly restrictive and premature
- The appropriate property with respect to arbitrage is from the perspectives outlined here, the martingale marginal property.
 - As noted earlier, here we seek martingales in other filtrations with the same marginal densities.
 - This is a much larger class of models than the martingale models.

Conclusion

- There are many potential applications for the resulting fitted *SVLP* models as in the main one may effectively summarize the surface quite parsimoniously with just 6 parameters as for example with *VGSA*.
- The six parameters have good economic interpretation in terms of risk premia for direction and size that one may back out of the Lévy density.
- One may also use the fitted model to generate via the Dupire equation the surface of local volatilities.
 - Here we present local volatilities for December 13 2000 from *VGSA* in accordance with

$$\sigma^2(K, T) = \frac{2(C_T + (r - q)KC_K + qC)}{K^2C_{KK}}$$

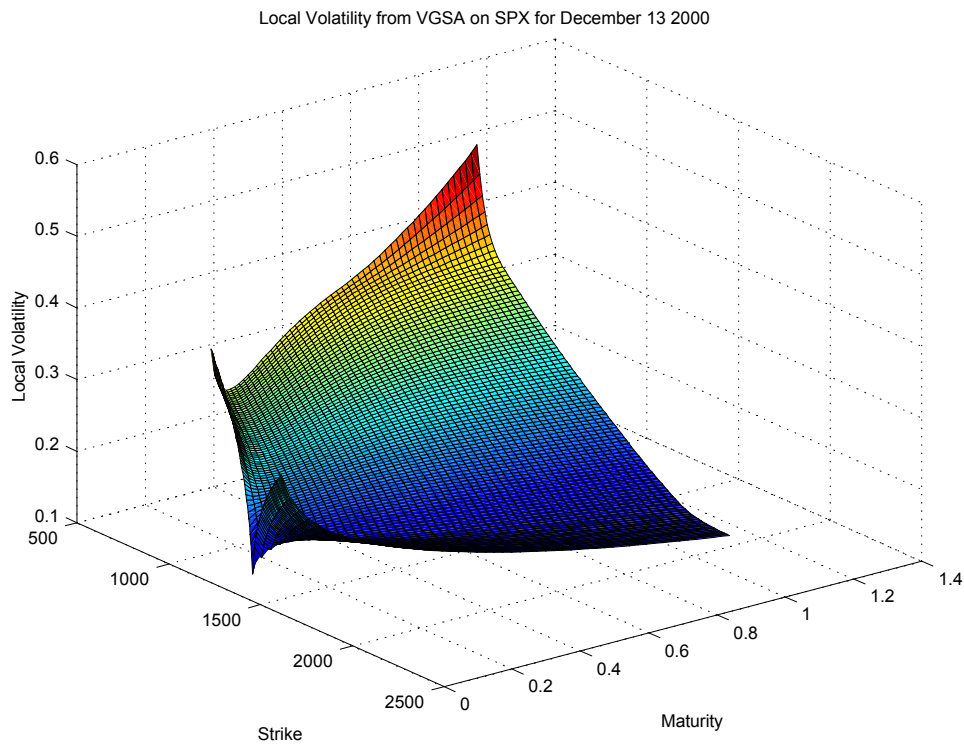


Figure 6: Local Volatility from VGSA on SPX for December 13 2000

- Barrier options may be priced using both *VGSA* and local volatility model derived from *VGSA*.