Asset Prices are Brownian motion: only in Business Time.*

Helyette Geman
University Paris IX Dauphine and ESSEC

Dilip B. Madan
University of Maryland

Marc Yor
University Paris VI - Laboratoire de Probabilités

October 27, 1998

Abstract

This paper argues that asset price processes arising from market clearing conditions should be modeled as pure jump processes, with no continuous martingale component. However, we show that continuity and normality can always be obtained after a time change. We study various examples of time changes and show that in all cases they are related to measures of economic activity. For the most general class of processes, the time change is a size-weighted sum of order arrivals. The paper provides a number of new processes for modeling prices. Characteristic functions for these processes are also given in closed form.

*This paper was completed during the time Dilip Madan was visiting the Centre de Recherches pour l'Enseignement de la Gestion, at the University of Paris IX Dauphine.

1 Introduction

Continuity of price processes has served economic theory as a convenient and powerful assumption, delivering market completeness and unique pricing of contingent claims by arbitrage. Such assumptions are critical to the validity of the Black-Scholes (1973) and Merton (1973) option pricing theories and their associated dynamic hedging strategies, and they are also fundamental to the Cox and Huang (1989) approach to solving the Merton (1971) intertemporal consumption and investment allocation problem. The assumption of continuity justifies the Cox, Ross and Rubinstein (1979) binomial approximation of the process, using up and down price shocks tending to zero with the size of the time grid. We grant that such price processes can accurately represent market prices in economies that instantaneously and continuously equilibrate to information flows driven by *diffusions or Ito processes*. Examples of such equilibrium models in the literature include Duffie and Huang (1985), Dumas (1989), He and Leland (1993), and Detemple and Murthy (1994).

We question the validity of diffusions as an appropriate model for the underlying uncertainties. Instead, we represent the price process as instantaneously and continuously adjusting to exogenous demand and supply shocks. We view the underlying uncertainties (represented by cumulated shocks) as consisting of increasing random processes. As such, our price processes turn out to be the difference of two increasing random processes, representing respectively the up and down moves of the market. In contrast with Brownian motion, such processes are of finite variation. Furthermore, we demonstrate that in the presence of local uncertainty such processes are generally purely discontinuous.[1]

The implications of our theory for economic analysis and risk management are profound. At variance with the Black Scholes Merton setting, options can no longer be replicated by trading in the stock and money market account. In fact, the primary motivation for the use of continuous processes as representing price movements was not its accuracy, but rather the dynamic hedging argument, valid in this context, that made options redundant assets and led to the Hakansson (1979) paradox. In our economy, not only do we obtain potentially greater accuracy in describing stock price changes, but options become primary market completing assets useful in hedging jump risks

[1]An exception is provided by the local times of Brownian motion.

while option prices constitute a rich source of information to be employed in designing optimal risk exposures.

The possibility of discontinuities or jumps in asset prices has a long history in the economics literature. Merton (1976) considered the addition of a jump component to the classical geometric Brownian motion model for the pricing of options on stocks. In explaining the distribution of returns such models were used as early as Press (1965). A number of authors have since considered the issue of market completeness in this context (see for instance Jones (1984)), Jarrow and Madan (1995)), while others have assessed and argued for their necessity in explaining implied volatility smiles in low maturity options (see for instance Bates (1996), Bakshi, Cao and Chen (1997)). However, these models all contain a diffusion component in addition to a low or finite activity jump part: the diffusion component accounts for high activity in price fluctuations while the jump component is used to account for rare and extreme movements.

By contrast, we account for the small, high activity and rare large moves of the price process in both a unified and connected manner. For the processes we consider, price jumps are the rule and all motion occurs via jumps. High activity is accounted for by a large (in fact infinite) number of small jumps. The activity at various jump sizes is analytically connected by the requirement that smaller jumps occur at a higher rate than larger jumps.

The property of an infinite number of small moves is shared with the diffusion-based models, with the additionally attractive feature that the sum of absolute changes in price is finite for our processes while for diffusions this is infinite (for diffusions, the price changes must be squared before they sum to a finite value). This makes it possible for us to create and price contracts based on the instantaneous upward, downward or total variability (positive, negative, or absolute price jump size) of underlying asset prices, in addition to the more traditional contracts with payoffs that are functionally related to the level of the underlying price. Processes similar to ours have recently been used in option pricing models by Heston (1993), Madan and Milne (1991), and include the $\alpha-$ stable increments for $\alpha < 1$ that were studied by Mandelbrot (1963) and McCulloch (1978).

Though the processes we advocate are pure jump processes and of finite total variation, we generalize the results of Clark (1973) and show that they may always be viewed as continuous processes in an economic measure of time. Clark (1973) considered subordinated processes where prices were represented by a geometric Brownian motion and time was given by another independent geometric Brownian motion. The processes we propose can also be written as Brownian motion evaluated at a random time change (or a stochastic clock), but in general the process of the clock need not be independent of the price process. We study the relationship between the stochastic clock and the demand and supply shocks driving the price process. As in Ané and Geman (1997) who show empirically that a quasi-perfect normality of returns on a high frequency data base of FTSE100 futures prices can be recovered using a stochastic clock driven by the number of trades, the time change we consider in our different examples is related to a measure of economic activity. We observe that price continuity does arise, but in an activity-based measure of time, as opposed to calendar time directly; hence the title of the paper.

The literature relating price changes to measures of activity (see for examples, Tauchen and Pitts (1983), Karpoff (1987), Gallant, Rossi and Tauchen (1992), and Jones, Kaul and Lipson (1994)) has considered as relevant measures of activity either the number of trades or the volume. Our analytical results on the nature of the time change show that both the number and size of orders enter the time calculation in specific ways. The precise time changes relevant for various example economies suggests a measure of time based on a size weighted total orders. Furthermore, the time change generally belongs to the *same class* of processes as the process for the underlying activity.

The analysis of our paper provides a useful reduced form synthesis of continuous price equilibrium processes. First we note that equilibrium processes are free of arbitrage opportunities. A rich literature has studied the consequences of the no arbitrage assumption (Harrison and Kreps(1979), Kreps (1981), Harrison and Pliska (1981)) and has concluded that such processes are semimartingales (Delbaen and Schachermayer (1994)). Monroe (1978) characterized all semimartingales and showed that every semimartingale may be represented as a time-changed Brownian motion. A consequence of these results taken jointly is that equilibrium price processes may be identified by focusing attention on the Brownian motion and its time change. We note that time changes are increasing processes and argue that when they are locally uncertain, they must be pure jump processes. We relate the time change to the information provided by demand and supply shocks in the market. [2]

[2]The resulting jump process need not be one of bounded variation and it may not be capable of being decomposed into the difference of two increasing processes if the infinite activity of small changes is too large; however, we view the hypothesis of infinite variation as unrealistic and consider mainly processes of bounded variation as good pricing models.

We first consider examples where the time change is independent of the Brownian motion (which was the assumption made in Clark (1973)). The identification of the time change in this context is easier. However, we also study more generally, examples where this independence assumption is relaxed (we note that the theorem of Monroe (1978) does not involve this property). In the case where there is dependence in our example economies, we show that the time change is cumulated volatility, where the latter depends on the Brownian motion. For a large class of processes both representations are valid: they can be written as a Brownian motion evaluated at an independent time, and also as another Brownian motion evaluated at another, now dependent time. We consider both representations as they provide a richer source of interesting and tractable models.

The two formulations are similar, in that in both price changes are related to excess demand. In the examples involving independence, excess demand coincides with orders as we assume no coincidence of demand and supply, and the price change is linearly related to the excess demand. In the second situation excess demand is modeled by a Brownian motion and the price response may be nonlinear in the excess demand. The model primitives for the first formulation are the arrival rates of buy and sell orders, while for the second formulation it is the function relating price responses to excess demand.

In many cases we derive closed form expressions for the characteristic function of the log price relative. It is well known how one may then extract the density or the distribution function via Fourier inversion from this characteristic function. Bakshi and Madan (1997) show how one may use this characteristic function to obtain option prices.[3] Hence, our analysis provides a wide class of operational models for continuous time price processes in economics. With the advent of price discovery in many previously regulated markets, and the opening of derivative markets for a much greater range of underlying processes, it is imperative that we expand the class of processes for which estimation and derivative pricing are feasible. This paper makes substantial contributions in this direction.

In any case, the infinite variation jump processes can always be approximated by the difference of two increasing processes, even if they are not identical to it.

[3]The difficulty addressed in Bakshi and Madan (1997), is that of deriving the characteristic function of a probability element constructed to correspond to the term $N(d_1)$ in the Black Scholes formula. For many cases this probability element is not in the same parametric class as the equivalent of $N(d_2)$.

The outline of the paper is as follows: Section 2 presents a general economic model for the process of price discovery in market economies. It is observed that this process is purely discontinuous and of finite variation. Section 3 shows how in general one may recover continuity of prices in a time given by a stochastic clock. Section 4 considers specific examples, where we relate the stochastic time to the uncertainties driving the price process and observe that this time is as an activity based measure. In Section 5 we consider some time changes that have been proposed in the literature, and are not related to the process of price discovery described in section 2, but can be approximated by such processes. Section 6 concludes.

2 The Price Discovery Process

Consider an economy over the time interval [0,T] in which a single commodity is traded continuously in time and has a price process, denoted $p(t)$, $0 \leq t \leq T$. The uncertainties driving the price process are identified as exogenous events representing demand or supply shocks respectively. Let $U(t)$ be the process of cumulated demand shocks, given by an increasing pure jump process. At any time t,

$$u_t = \Delta U(t) = U(t) - U(t_-) \geq 0,$$

represents the number of units of the asset that are demanded by agents in the economy consequent upon the occurrence of a liquidity or information based event. The quantity u_t is the amount demanded at the prevailing price of $p(t_-)$ and in the absence of any price response to the processing of the demand order. We think of u_t as the amount some economic agent would like to buy at the price $p(t_-)$. The considerations determining u_t are either liquidity considerations, wealth or cash balance accumulations of individuals or information considerations reflecting beliefs of individuals that the asset is underpriced at $p(t_-)$. $U(t)$ is the cumulated level of such demand shocks that are motivated by liquidity or information shocks. Similarly, let $V(t)$ be the cumulated level of such supply shocks or amounts that agents would like to sell at the price $p(t_-)$. The supply shock is

$$v_t = \Delta V(t) = V(t) - V(t_-).$$

Assumption 1: Non-coincidence of Demand and Supply Shocks in Continuous Time

We suppose that cumulated demand shocks, $U(t)$, and cumulated supply shocks, $V(t)$, are increasing pure jump processes with no coincidence of jumps in continuous time.

The primary sources for these shocks are the arrival of information in the economy that is transformed into market buy and sell orders and we view these primary sources of uncertainty as exogenous. We recognize that differences in beliefs among market participants can lead to simultaneous buy and sell orders triggered by a single information event, but suppose that there are sufficient differences in the access to the information, and the subsequent execution decisions, for the actual orders to materialize at different instants of continuous time. Under this assumption the markets are dealing, at all instants, with either one or the other type of order.

2.1 Modeling the Price Increases

All market participants realize that orders won't be executed at the price $p(t_{-})$ and expect a price response to the execution of orders.[4] We suppose that the actual quantity demanded in response to a demand shock u_t , q_t^{du} at time t , is given by a demand function

$$q_t^{du} = q^{du}(p(t)/p(t_{-}), u_t, t) \tag{1}$$

that reflects a falling off of demand in the face of a price response. The quantity u_t is, as stated earlier, the amount an economic agent would like to buy at the price $p(t_{-})$, and (1) gives the agent's demand response to price increases. We suppose that

$$\frac{\partial q^{du}}{\partial p(t)} < 0.$$

[4] We recognize that small orders are not likely to experience an adverse price impact and accomodate this by elasticity assumptions on demand and supply for low quantities. We also recognize that incentives for order splitting in the presence of low price responses to small orders can lead to strategies of submitting an infinite number of infinitesimal orders to complete finite transactions, but rely on the exogeneous costs associated with such splitting, to rule out these possibilities.

Furthermore, we have that u_t is the demand at no price response or that:

$$q^{du}(1, u_t, t) = u_t.$$

In addition to the demand function, there is a supply function with suppliers being aware of the liquidity or information event and the extent of this exogenous demand shock. The supply is given by

$$q_t^{su} = q^{su}(p(t)/p(t_-), u_t, t) \tag{2}$$

where we suppose that

$$\frac{\partial q^{su}}{\partial p(t)} > 0$$

and furthermore we assume no supply at a zero price response or that:

$$q^{su}(1, u_t, t) = 0.$$

Under these conditions all supply occurs at a price response.[5] Equation (2) may also be viewed as reflecting the price response in a limit order sell book on the market, though more generally it is the supply function of the economy that is relevant.

The market price $p(t)$ and quantity transacted q_t^u are simultaneously determined in equilibrium by the market clearing condition

$$q_t^u = q_t^{du} = q_t^{su}. \tag{3}$$

We suppose that the market clearing condition (3) may be solved to determine simultaneously the price response

$$\ln\left(\frac{p(t)}{p(t_-)}\right) = \Phi^u(u_t, t) > 0 \tag{4}$$

and the quantity transacted

$$q_t^u = \Psi^u(u_t, t) > 0. \tag{5}$$

We may accommodate very small price responses to small shocks by supposing that the supply elasticity is high for small quantities.

[5]The supply curve q^{su} represents supply at prices higher than $p(t_-)$. Supply at $p(t_-)$ comes forth as a market sell order, at instants that differ from those at which buy orders are coming to market.

It is useful to contrast this formulation with the more standard general equilibrium formulation where at each instant, all markets are simultaneously cleared to determine all asset prices. In our partial equilibrium model, the market clearing takes place in one market at a time. Our demand and supply curves are contingent on the arrival of a demand order for this asset at time t, in the quantity u_t if the price stays at $p(t_{-})$.

2.2 Modeling the Price Decreases

For price decreases we follow a symmetric approach to that taken in modeling price increases. In the presence of a supply shock v_t at time t , to sell v_t if the price stays at $p(t_{-})$, the supply function

$$q_t^{sv} = q^{sv}(p(t)/p(t_{-}), v_t, t) \tag{6}$$

reflects the curtailment of supply in the presence of a downward price response and we suppose that

$$\frac{\partial q^{sv}}{\partial p(t)} > 0$$

while the supply is v_t at no price response, or that:

$$q^{sv}(1, v_t, t) = v_t.$$

The demand function in the face of an exogenous supply shock is given by

$$q_t^{dv} = q^{dv}(p(t)/p(t_{-}), v_t, t) \tag{7}$$

and reflects the limit order buy book and more generally the economy wide demand response.We supose that

$$\frac{\partial q^{dv}}{\partial p(t)} < 0$$

and furthermore that there is no demand at a zero price response,

$$q^{dv}(1, v_t, t) = 0.$$

Once again, the partial equilibrium condition in the presence of a supply shock determines the quantity supplied q_t^v and the condition is

$$q_t^v = q_t^{sv} = q_t^{dv}. \tag{8}$$

We suppose that the market clearing conditions may be solved for the price responses and the quantity transacted to yield

$$\ln\left(\frac{p(t)}{p(t_{-})}\right) = -\Phi^v(v_t, t) < 0 \tag{9}$$

and

$$q_t^v = \Psi^v(v_t, t) \tag{10}$$

Note that $\Phi^v(v_t, t)$ is the absolute value of the log price response and will later be used to cumulate the total price decreases. We may accommodate small price responses for small shocks by supposing that the economy wide demand elasticity is high for small quantities.

2.3 The Price Process

In this section we put together the results for the price increases and decreases and derive the complete price process. Under assumption 1 the demand and supply shocks never arise at the same instance of time and hence the price process is given by

$$\ln(p(t)) = \ln(p(0)) + \sum_{s\leq t}\Phi^u(\Delta U(s), s) - \sum_{s\leq t}\Phi^v(\Delta V(s), s) \tag{11}$$

while the process for the total quantity transacted to time t, is given by $Q(t)$, where

$$Q(t) = \sum_{s\leq t}\Psi^u(\Delta U(s), s) + \sum_{s\leq t}\Psi^v(\Delta V(s), s). \tag{12}$$

We observe from equation (12) that the process for the volume of transactions to time t is the sum of two increasing pure jump processes

$$Y_1(t) = \sum_{s\leq t}\Psi^u(\Delta U(s), s) \tag{13}$$

and

$$Y_2(t) = \sum_{s\leq t}\Psi^v(\Delta V(s), s). \tag{14}$$

It follows that $Q(t)$ is a pure jump process of finite variation and the volume transacted between two time points $t_1 < t_2$, $V(t_1, t_2)$ may be recovered as

$$V(t_1, t_2) = Q(t_2) - Q(t_1).$$

Similarly, we observe from equation (11) that the price process is the difference of two increasing pure jump processes

$$X_1(t) = \sum_{s \le t} \Phi^u(\Delta U(s), s) \tag{15}$$

and

$$X_2(t) = \sum_{s \le t} \Phi^v(\Delta V(s), s). \tag{16}$$

It follows that $\ln(p(t))$ *is a pure jump process of finite variation.*[6]

This formulation of the price process is in sharp contrast to traditional assumptions about such processes in the finance literature. Typically it is assumed that asset prices follow a diffusion process, and as a result, are continuous processes of infinite variation. Duffie and Huang (1985) provide a general existence theorem for economies in which equilibrium price processes are diffusions. He and Leland (1993) consider conditions on price processes in the diffusion context, that ensure their consistency with an economic equilibrium.

The motivation for our departure from these more traditional formulations lies in the modeling of the underlying uncertainties. We view the underlying uncertainties as consisting of increasing random processes. These could be processes for total prevailing price buy orders, and total prevailing price sell orders, viewed separately. Net orders, being the difference, are then by construction a process of bounded variation and cannot be accurately modeled by a continuous diffusion, which is of infinite variation. Many authors assume that the underlying uncertainty is a continuous diffusion and this then implies the same for the price process via the market equilibrium conditions.

[6] A process $x(t)$ is said to be of finite variation over the interval $[0, T]$ if for any path, there exists a constant A, such that for all partitions $0 = t_0 < t_1 < \cdots < t_n = T$ we have $\sum_{i=0}^{n-1} |x(t_{i+1}) - x(t_i)| < A$. A process of finite variation can be written as the difference of two increasing processes. The converse is also true.

2.4 Parameterizing the Price Process

We now introduce specific functional forms for the demand and supply functions that permit a closed form parametrization of the price process.

Assumption 2: Parametric Responses to Demand Shocks
Suppose that the demand function q_t^{du} is given by

$$q_t^{du} = u_t - \delta_t \ln\left(\frac{p(t)}{p(t_-)}\right) \tag{17}$$

while the supply function in the face of a demand shock is

$$q_t^{su} = a_t u_t^{-\gamma_t} \ln\left(\frac{p(t)}{p(t_-)}\right) \tag{18}$$

reflecting a lower supply if the demand shock is large for positive γ_t.

Note that the response functions (17) and (18) meet the requirements that at $p(t) = p(t_-)$, demand is u_t while supply is 0, as is required by the interpretation of u_t.

Assumption 3: Parametric Responses to Supply Shocks
We suppose that in the presence of a supply shock we have

$$q_t^{sv} = v_t + \eta_t \ln\left(\frac{p(t)}{p(t_-)}\right) \tag{19}$$

while

$$q_t^{sd} = -b_t v_t^{-\lambda_t} \ln\left(\frac{p(t)}{p(t_-)}\right) \tag{20}$$

and the demand is curtailed for large supply shocks for positive λ_t.

Solving for the equilibrium using equations (17) and (18) we obtain that

$$\Phi^u(u_t, t) = \frac{u_t}{\delta_t + a_t u_t^{-\gamma_t}} \tag{21}$$

while

$$\Psi^u(u_t, t) = \frac{a_t u_t^{1-\gamma_t}}{\delta_t + a_t u_t^{-\gamma_t}} \tag{22}$$

Similarly, the equilibrium solutions using equations (19) and (20) are

$$\Phi^v(v_t, t) = \frac{v_t}{\eta_t + b_t v_t^{-\lambda_t}} \tag{23}$$

and

$$\Psi^v(v_t, t) = \frac{b_t v_t^{1-\lambda_t}}{\eta_t + b_t v_t^{-\lambda_t}} \tag{24}$$

Under these parametric assumptions one may write the price process as

$$\ln p(t) = \ln p(0) + \sum_{s \le t} \frac{\Delta U(s)}{\delta_s + a_s(\Delta U(s))^{-\gamma_s}} - \sum_{s \le t} \frac{\Delta V(s)}{\eta_s + b_s(\Delta V(s))^{-\lambda_s}} \tag{25}$$

and the process for cumulated transactions is

$$Q(t) = \sum_{s \le t} \frac{a_s \Delta U(s)}{a_s + \delta_s(\Delta U(s))^{\gamma_s}} + \sum_{s \le t} \frac{b_s \Delta V(s)}{b_s + \eta_s(\Delta V(s))^{\lambda_s}} \tag{26}$$

Assumption 4: Stationary Limit Order Books

The special case of stationary limit order books occurs when the supply responses to market order demand shocks and the demand responses to market sell shocks are not responsive to the size of these shocks. In this case γ_s, λ_s are zero and the coefficients of the demand and supply functions are constant through time, $\delta_t = \delta, a_t = a, \eta_t = \eta$, and $b_t = b$.

Under assumptions 1 through 4, we may write the price process as

$$\ln p(t) = \frac{1}{\delta + a} U(t) - \frac{1}{\eta + b} V(t) + \ln p(0) \tag{27}$$

while

$$Q(t) = \frac{a}{\delta + a} U(t) + \frac{b}{\eta + b} V(t). \tag{28}$$

Under these assumptions the price process is the difference of two increasing pure jump processes while the transacted quantity is related to the sum of these processes. Furthermore, we have linearity of the price response in the order flow. Later we do consider models that allow for nonlinearities in the price response to excess demand.

3 Stochastic Time Changes and Prices

It was argued in section 2 that in general the process for the price of an asset is the difference of two increasing pure jump processes and is a process of finite variation. In this section we observe, interestingly, that continuity of the price process can always be recovered by the use of a stochastic time change. In effect, there is always a sense of economic time, related to market activity, such that prices are continuous when time is measured in units of business activity. In the next section we begin our study of the relationship between this time change and the original price process.

The price process of section 2, by virtue of being of bounded variation, is a semimartingale. It is also known that every price process that is consistent with the absence of arbitrage opportunities, must be a semimartingale (Delbaen and Schachermayer (1994)). Hence all equilibrium price processes meet this condition, whether they are diffusions or not. The next question is, 'What semimartingales are most appropriate as models for price processes?' For this we turn to Monroe (1978).

A remarkable result of Monroe (1978)[7] shows that every semimartingale can be written as a Brownian motion (possibly defined on some adequately extended probability space) evaluated at a random time.[8] By this result there exists a Brownian motion $W(t)$ and a random time change $T(t)$, where $T(t)$ is an increasing stochastic process[9], such that

$$\ln(p(t)) = \ln(p(0)) + W(T(t)). \tag{29}$$

[7]We wish to thank Joe Horowitz for bringing this result to our attention.

[8]It is instructive to note that one may even write calendar time as time changed Brownian motion, whereby

$$t = W(T(t))$$

where $T(t)$ is defined by

$$T(t) = \inf\{s|W(s) \geq t\}.$$

One clearly observes the dependence between the time change and Brownian motion in this case. We also observe from (**??**) that for any increasing process $A(t)$

$$A(t) = W(T(A(t)) = W(T^*(t)),$$

where $T^*(t)$ is another time change.

[9]More formally, it is required that there exists a filtration with respect to which the process $W(t)$ is adapted, and that $T(t)$ is an increasing sequence of stopping times adapted to this filtration.

Equation (29) implies that the study of price processes for market economies may be *reduced to the study of time changes* for Brownian motion. This is a powerful reduced form representation of a complex phenomenon involving multidimensional considerations - those of modeling demand, supply and their interaction through market clearing - to a single entity: the correct unit of time for the economy, with respect to which we have a Brownian motion.

We note that the price process will be continuous in calendar time only if the time change is continuous. This observation has some restrictive implications. If the time change is continuous it must essentially be a stochastic integral with respect to Brownian motion (Revuz and Yor (1994), page 190 Theorem 3.9).[10] For a time change, it follows that the diffusion component must be zero: otherwise it will not be an increasing process and so cannot be a time change. We therefore have the implication that continuous time changes are locally deterministic. We summarize this discussion in the following proposition.

Proposition 1 *In the absence of arbitrage opportunities, continuous time price processes of market economies are time-changed Brownian motions. Furthermore, if there is local uncertainty in the time change then the price process is not a continuous process in any interval of time.*

In addition, proposition 1 suggests that return distributions should be normal, not when measured in calendar time, but when measured per unit of what may be termed, economic time. The search for such a formulation began in earnest with Clark (1973), and has been pursued more recently by Ané and Geman (1997) and Madan and Chang (1998). In fact, it is shown in Ané and Geman (1997) that one minute, FTSE100, calendar time returns are highly non-normal, while returns measured per unit trade are normally distributed. Madan and Chang (1998) show that Brownian motion, when time changed by a gamma process, provides a significantly better description of historical asset returns and the risk neutral return distribution embedded

[10]The qualification required is that its quadratic characteristic is absolutely continuous with respect to Lebesgue measure or equivalently that it has a well defined sense of a variance rate. As noted earlier, the local time of Brownian motion is an example of a continuous increasing random process which has intervals of time when the clock does not move at all, and is hence not appropriate for our consideration. We restrict attention to processes with a well defined sense of a realized variance rate.

in option prices. Our investigations here will focus on theoretically identifying and interpreting $T(t)$ from a knowledge of the process $\ln(p(t))$ as a process of bounded variation.[11] In particular we wish to explore the relationship between time changes and general measures of economic activity.

We first consider representations of the type given by equation (29) in which the Brownian motion is independent of the time change. Furthermore, to allow for additional generality we consider stochastic time changes applied to continuous Ito processes. From this perspective, let $x(t)$ be an Ito process of the general form

$$x(t) = x(0) + \int_0^t \theta(u)du + \int_0^t \sigma(u)dW(u). \tag{30}$$

We consider the representation of the price process as $x(t)$ evaluated at a random time $T(t)$ or

$$\ln(p(t)) = \ln(p(0)) + x(T(t)), \tag{31}$$

where $T(t)$ is independent of $(x(u), u \geq 0)$.

Some general properties of the time change $T(t)$ are inherited from the price process. Firstly, as already noted, since the price process is pure jump, and $x(t)$ is continuous, it follows that $T(t)$ is a pure jump process. Second, we say that the price process represents a high level of activity if there is no interval of time in which prices are constant throughout the time interval. Processes of high activity levels have an infinite arrival rate of demand and supply shocks, though of necessity the arrival rate is finite for all shocks with a magnitude strictly bounded away from zero. If the price process is such a high activity process, then this property is also inherited by the time change. In the next section we explore examples where the time change, as a process, reflects quite exactly the two processes for the demand and supply shocks.

[11] We recognize that semimartingales may be of infinite variation and so the price process may not be of bounded variation, but note that even in this case, an approximation by bounded variation processes is admissible. Note for example that the usual Binomial approximation to Brownian motion is a process of bounded variation.

4 Time Changes related to Demand and Supply Shocks

This section studies the relationship of stochastic time changes to the process of demand and supply shocks, in some simple and tractable contexts. We begin with processes that have a finite arrival rate and then consider high activity processes with infinite arrival rates. We study these relationships between time changes and demand and supply shocks in the context of various examples for the price process.

We first consider two pure jump cases with finite jump arrival rates and jump sizes distributed as i) a reflected normal distribution, as is the case with the Merton (1976) jump diffusion model, and ii) for exponentially distributed jump sizes arriving at Poisson times, as this class of processes is a base model for the construction of a wide class of increasing random processes and time changes. In the latter case we have jump arrival rates exponentially related to the jump size. For these cases we show that in i) the number of arrivals, independent of the size, constitutes the time change, but in ii) arrivals of different sizes have differing impacts on the stochastic clock. We show that the relevant time measure is a size weighted cumulation of order arrivals.

Our next example considers the case of the gamma process that turns out to be a fundamental building block for a wide class of infinite arrival rate candidate processes. We show that for the gamma process a proxy for the time change over an interval is related interestingly to the level of excess demand. This is reasonable as demand matched by supply does not contribute to price pressure.

The result for gamma processes is next generalized to a wide class of increasing random processes that have a desirable structure on the arrival rate of jump sizes, which is that larger jumps have a smaller arrival rate. The relationship between the level of economic activity and the time change generalizes the result obtained for the compound Poisson process with exponentially distributed jump sizes.

We next investigate the similarity of the time change and the order arrival processes. In particular we observe that when the order arrival processes are in a given subclass of processes, the time change also belongs to the same subclass.

An important example of a subclass is given by processes for which larger jump, size weighted arrival rates, are smaller. This is the class of processes that can be generated from the gamma process by convolution, termed the class of generalized gamma convolutions. Once again the time change is also a generalized gamma convolution. A useful generalization of the variance gamma model in the generalized gamma convolution family of processes is also provided.

The next subsection considers this question of similarity of order arrival and time change for the case of the stable processes. It is observed that for $\alpha-$stable arrival rate processes the time change is a stable process with index $\alpha/2$.

We then ask when the time change is a simple adjustment of scale and speed of the process for the arrival of shocks. This class consists of the gamma process and processes constructed from the gamma process by integrating the tail of the gamma arrival rates, as is made precise later. Interestingly, the gamma process itself is a consequence of applying this procedure to the compound Poisson process with exponentially distributed jump sizes.

For processes with completely monotone arrival rates as a function of the jump size[12], we present an alternative economic representation of the process. We model excess demand by a Brownian motion and define a *force function* that expresses the derivative of prices as a function of this excess demand. The equilibrium price process is given by restricting attention to the zero excess demand situations or evaluating prices at the inverse local time of Brownian motion, where by construction Brownian motion is zero and we have equilibrium. We establish the connections between the force function and the arrival rates of price moves, providing in addition various examples relating these two constructs.

All the processes for the log price relative $X(t) = \ln(p(t)/p(0))$ considered in this section are pure jump processes with independent and identically distributed increments over non-overlapping intervals of regular length. We recognize the considerable econometric evidence in support of time inhomogeneity in both the statistical and risk neutral price processes, but focus attention here primarily on the correct characterization of the local motion. For this purpose, time homogeneous processes, like Brownian motion itself, are an appropriate starting point for the discussion. Generalizations incorporating time inhomogeneity can be accommodated later as appropriate for the

[12]A real valued function of one variable is monotone if its derivative does not change sign. It is completely monotone, if all its derivatives have this property. With respect to arrival rates we require that their derivatives with respect to the jump size alternate in sign, begining with negative for positive jumps and positive for negative jumps.

application at hand. The processes we consider are completely characterized by their Lévy densities $k(x)-$ defining the arrival rates of price jumps of size $x-$ that are identified by the unique decomposition of their characteristic functions provided by the Lévy Khintchine theorem that asserts that [13]

$$E\left[\exp\left(iuX(t)\right)\right] = \exp\left(t\int_{-\infty}^{\infty}\left(e^{iux}-1\right)k(x)dx\right). \tag{32}$$

In each case we seek to rewrite the process $X(t)$ as a time-changed Brownian motion. If we consider Brownian motion with drift

$$B(t) = \theta t + \sigma W(t) \tag{33}$$

and evaluate this at an independent random time $T(t)$, with characteristic function given by the Lévy measure $\widetilde{k}(x)$, where

$$E\left[\exp\left(iuT(t)\right)\right] = \exp\left(t\int_{0}^{\infty}\left(e^{iux}-1\right)\widetilde{k}(x)dx\right). \tag{34}$$

We seek to discover the relationship between k and $\widetilde{k}$ under the condition that $X(t)$ may be written as Brownian motion time changed by $T(t)$, or that

$$X(t) = B(T(t)).$$

From the independence of the Brownian motion and the time change one may infer that

$$\begin{aligned} E\left[\exp\left(iuB(T(t))\right)\right] &= E\left[\exp\left(iu\theta T(t) + \sigma W(T(t))\right)\right] \\ &= E\left[\exp((iu\theta - \sigma^2u^2/2)T(t))\right] \\ &= \exp\left(t\int_{0}^{\infty}(e^{(iu\theta-\sigma^2u^2/2)x}-1)\widetilde{k}(x)dx\right) \\ &= \exp\left(t\int_{-\infty}^{\infty}\left(e^{iux}-1\right)k(x)dx\right) \end{aligned} \tag{35}$$

[13] We suppose both the absence of a continuous martingale component and a deterministic drift rate. A deterministic drift rate may easily be added in applications, we are here concerned with just the stochastic component.

The final equality in (35) follows from (32) on noting that $B(T(t))$ is also $X(t)$. In each case of independence of the Brownian motion and the time change, the Lévy measure for the time change $\widetilde{k}$ is discovered given the arrival rate Lévy measure by solving for $\widetilde{k}$ in the last equality of (35). In the absence of independence the construction is more complicated and we consider this in subsection 4.8.

4.1 Compound Poisson Demand and Supply Shocks

This subsection considers two finite arrival rate processes with arrivals occurring at Poisson times. The first case is one that mirrors the Merton (1976) jump diffusion model and considers a reflected normal distribution for the jump sizes. In the second case we consider exponentially distributed jump sizes. We shall note later that the first case is not completely monotone, while the second is the building block for this family of processes.

4.1.1 Reflected Normal Prevailing Price Order Sizes

Let the prevailing price buy and sell orders be given by independent copies of the increasing compound Poisson process

$$X(t) = \sum_{i=1}^{N(t)} Y_i \tag{36}$$

where $N(t)$ is a Poisson process with arrival rate λt and the sequence of the magnitude of demand and supply shocks Y_i are independently and identically distributed with a reflected normal density,

$$f(y) = \frac{\sqrt{2}\exp\left(-\frac{y^2}{2\sigma^2}\right)}{\sigma\sqrt{\pi}}, \text{ for } y > 0. \tag{37}$$

In this example the time change to be applied to Brownian motion to recover the price process turns out to be the total number of prevailing price orders, buy or sell.

Suppose that $\ln(p(t))$ is given by equation (27) with $(\delta+a)^{-1} = (\eta+b)^{-1} = \gamma$. In this case we may write

$$\ln(p(t)/p(0)) = \gamma(X_1(t) - X_2(t)) \tag{38}$$

where $X_1(t), X_2(t)$ are two independent copies of the process satisfying (36).

Let $\phi_Y(u)$ be the characteristic function of Y , so that

$$\phi_Y(u) = \int_0^\infty \exp(iuy) \frac{\sqrt{2}\exp\left(-\frac{y^2}{2\sigma^2}\right)}{\sigma\sqrt{\pi}} dy. \tag{39}$$

Since the order size is positive or negative with equal probability, the characteristic function of the order size is given by

$$\mathrm{Re}(\phi_Y(u)) = \exp\left(-\frac{\sigma^2 u^2}{2}\right).$$

It follows by direct computation of characteristic functions for compound Poisson processes (See for example Karlin and Taylor (1981)) that

$$\phi_{\ln(p(t)/p(0))}(u) = \exp\left(2\lambda t\left(\exp\left(-\frac{\sigma^2\gamma^2u^2}{2}\right) - 1\right)\right) \tag{40}$$

Equation (40) , however, is also the characteristic function of $\sigma\gamma W(N_1(t) + N_2(t))$, where $W(t)$ is a standard Brownian motion and $N_1(t), N_2(t)$ are two independent copies of a Poisson process with arrival rate λt.

In this example the time change just counts the number of all the demand and supply shocks, ignoring the magnitude of the shocks, which are accounted for by the distribution of the Brownian motion between successive arrivals. The volatility is of course scaled by γ to reflect the price sensitivity of the shocks. It is interesting to note that the time change of this example is akin to the number of trades, the time change observed to be relevant by Ané and Geman (1997) in their empirical study of high frequency returns on the FTSE100 futures index.

4.1.2 Exponential Jump Sizes

Suppose now that the prevailing price order sizes have an exponential density

$$f(y) = a\exp(-ay), \text{ for } y > 0 \tag{41}$$

with mean order size given by $1/a$. Further suppose that the Poisson arrival rate of jumps is $1/a$, so that the Lévy measure for the buy or sell orders is just the exponential function

$$k(x) = \exp(-ax) \tag{42}$$

Let $X_1(t)$ and $X_2(t)$ be two independent copies of this exponential compound Poisson process and suppose that log prices evolve in accordance with (38). The characteristic function for the log price relative is now easily computed as

$$\phi_{\ln(p(t)/p(0))}(u) = \exp\left(\frac{2t}{a}\left(\frac{a^2}{a^2+u^2\gamma^2} - 1\right)\right). \tag{43}$$

To observe this process as a time-changed Brownian motion consider the time change given by $T(t)$, an increasing process with Lévy measure

$$\widetilde{k}(y) = 2a\exp(-a^2 y), \text{ for } y > 0. \tag{44}$$

Now consider the process $\gamma\sqrt{2}W(T(t))$ and evaluate its characteristic function using (34) as

$$\begin{aligned} E\left[\exp\left(iu\gamma\sqrt{2}W(T(t))\right)\right] &= E\left[\exp\left(-\gamma^2 u^2 T(t)\right)\right] \\ &= \exp\left(t\int_0^\infty (e^{-\gamma^2u^2y} - 1)2ae^{-a^2y}dy\right) \\ &= \exp\left(2ta\left(\frac{1}{u^2\gamma^2+a^2} - \frac{1}{a^2}\right)\right) \\ &= \exp\left(\frac{2t}{a}\left(\frac{a^2}{a^2+u^2\gamma^2} - 1\right)\right) \end{aligned} \tag{45}$$

We now investigate and comment on the relationship obtained in (45) between the time change and the order arrival process. An order of size y, be it buy or sell, arrives at the rate $2\exp(-ay)$. To construct the time change we have to weight the orders of different sizes differently; in fact, orders of size y amount to $a\exp(-a(a-1)y)$ units of time change. For $a = 1$ each order amounts to a unit of time change, as in the case of the reflected normal process, but for $a > 1$ (when most of the orders are small), the small orders count for a greater time change.The opposite holds for $a < 1$, in which case large orders count for greater time changes. The break even point of order sizes that count for a unit time change is given in both cases by $\ln(a)/(a(a-1))$.

Our next example considers two high activity processes for the prevailing price buy and sell order processes. In particular we consider for $U(t)$ and $V(t)$, two independent gamma processes with a common coefficient of variation.

4.2 Gamma process Demand and Supply Shocks

Suppose that the prevailing price buy order process, $U(t)$, is a gamma process with mean and variance rates μ_1, ν_1, respectively, while the prevailing price sell order process, $V(t)$, is also an independent gamma process of mean and variance rates μ_2, ν_2 respectively. It is useful to write these processes in terms of the standard gamma process with unit mean and variance rates.

Let $\gamma(t)$ denote the standard gamma process of unit mean and variance rate. The standard gamma process is a pure jump increasing random process with characteristic function

$$\phi_{\gamma(t)}(u) = \left(\frac{1}{1-iu}\right)^t \tag{46}$$

and Lévy measure

$$K_\gamma(x)dx = \frac{\exp(-x)}{x}dx \text{ for } x > 0 \tag{47}$$

that integrates to infinity, and hence we have a process that jumps infinitely often in any interval (or a high activity process), in sharp contrast to the processes considered in the last subsection.

One may write the prevailing price buy and sell order processes, $U(t)$ and $V(t)$, in terms of the standard gamma process by

$$U(t) = \frac{\nu_1}{\mu_1}\gamma(\frac{\mu_1^2}{\nu_1}t), \text{ while } V(t) = \frac{\nu_2}{\mu_2}\gamma(\frac{\mu_2^2}{\nu_2}t).$$

We suppose, as in Madan and Chang (1998), that the two processes for the prevailing price buy and sell orders share the same coefficient of variation $\kappa = \frac{\mu_1^2}{\nu_1} = \frac{\mu_2^2}{\nu_2}$. Under this assumption the price process of equation (27) may now be written, letting $\alpha_1 = (\delta + a)^{-1}\nu_1/\mu_1$ and $\alpha_2 = (\eta + b)^{-1}\nu_2/\mu_2$, as

$$\ln(p(t)/p(0)) = \gamma_1(t) - \gamma_2(t) = \alpha_1\gamma(\kappa t) - \alpha_2\gamma(\kappa t). \tag{48}$$

The characteristic function of $\ln(p(t)/p(0))$ is then easily evaluated, using (46) as

$$\begin{aligned}\phi_{\ln(p(t)/p(0))}(u) &= \left(\frac{1}{1-i\alpha_1 u}\right)^{\kappa t}\left(\frac{1}{1+i\alpha_2 u}\right)^{\kappa t} \\ &= \left(\frac{1}{1-i(\alpha_1-\alpha_2)u+\alpha_1\alpha_2 u^2}\right)^{\kappa t}.\end{aligned} \tag{49}$$

To represent this process as a time-changed Brownian motion, consider a Brownian motion with drift θ and volatility σ evaluated at the gamma time, $\gamma_3(t) = \gamma(\kappa t)$. Hence, let

$$Y(t) = \theta\gamma_3(t) + \sigma W(\gamma_3(t))$$

where $W(t)$ is a standard Brownian motion. The characteristic function of Y conditional on the gamma time is

$$\phi_{Y(t)|\gamma_3(t)}(u) = \exp\left(iu\theta\gamma_3(t) - \frac{\sigma^2 u^2}{2}\gamma_3(t)\right) \tag{50}$$

and the characteristic function of Y is

$$\phi_Y(u) = \left(\frac{1}{1 - iu\theta + \frac{\sigma^2 u^2}{2}}\right)^{\kappa t}. \tag{51}$$

A comparison of (51) and (49) shows that with $\theta = \alpha_1 - \alpha_2$ and $\sigma = \sqrt{2\alpha_1\alpha_2}$, the price process may be expressed in probability law as a Brownian motion with drift evaluated at the gamma time, $\gamma(\kappa t)$.

The time change $\gamma_3(t)$ is easily related to the individual gamma processes that were differenced and we observe that

$$\gamma_3(t) = (1/\alpha_i)\gamma_i(t), \text{ for } i = 1, 2. \tag{52}$$

which is a simple change of scale of the original order arrival rate processes. Later we inquire (in subsection 4.5) as to when we may expect the time change to be related to the original order arrival time by a simple change of scale and speed.

A different view of the relationship between the prevailing price buy and sell order arrival processes and the time change may be obtained by focusing attention on the prices at time $t = (2\kappa)^{-1}$, in which case

$$\ln\left(\frac{p(\frac{1}{2\kappa})}{p(0)}\right) = \alpha_1\gamma(1/2) - \alpha_2\gamma(1/2) \tag{53}$$

It is easily verified that the probability law of $2\gamma(1/2)$ is that of the square of a standard normal variate and hence that

$$2\ln\left(\frac{p(\frac{1}{2\kappa})}{p(0)}\right) = \alpha_1 B - \alpha_2 S \tag{54}$$

where $B = N_d^2, S = N_s^2$, and N_d, N_s are two independent normal random variables of zero mean and unit variance. It is shown in the appendix that the time change at this same point is

$$\gamma_3(1/2) = \left(\sqrt{B} - \sqrt{S}\right)^2 \tag{55}$$

We thus observe that the time change is related to the level of excess demand. This is a reasonable property in that buy orders matched by supply orders do not result in price pressures.

4.3 Demand and Supply Shocks as Monotone Lévy Processes

An important structural property of Lévy densities is that of monotonicity. One expects that jumps of larger sizes have lower arrival rates than jumps of smaller sizes. This property amounts to asserting for differentiable densities that the derivative is negative for positive jump sizes and positive for negative jump sizes.. Modeling the negative jumps symmetrically with the positive ones, we restrict the discussion to the Lévy density for the positive jumps.

The property of monotonicity may be strengthened to complete monotonicity by requiring derivatives of the same order to have the same sign. The mathematical definition, however, further requires the signs to be alternating. A completely monotone Lévy density is decreasing and convex, its derivative is increasing and concave and so on. Structural restrictions of this sort are useful in modeling discontinuous phenomena, given the wide class of choices that are otherwise available to model the Lévy density, which basically is any positive function that integrates the minimum of x^2 and 1. Complete monotonicity also has the interesting property of linking analytically the arrival rates of large jumps to that of small ones by requiring the latter to be larger than the former. The presence of such links makes it possible to learn about larger jumps from observing smaller ones.

In this regard we note that the jump diffusion model based on the reflected normal distribution for the jump sizes is not completely monotone as is easily observed by noting that the normal density shifts from being a concave function near zero to a convex function near infinity. On the other hand the exponentially distributed jump size is the foundation for all completely monotone Lévy densities. By Bernstein's theorem all completely monotone

Lévy densities are given by the Laplace transforms of positive measures on the positive half line, or that there exists a measure $\rho(da)$ such that

$$k(y) = \int_0^\infty e^{-ay}\rho(da) \tag{56}$$

Such Lévy densities have the useful interpretation that the economy is populated by individuals who submit prevailing price buy or sell orders with an exponential distribution with mean order rate $(1/a)$: the measure $\rho(da)$ is a measure of the number of orders per unit time of this mean level, with exponential size distribution. All completely monotone Lévy densities come from such an economy.

We consider now prevailing price buy orders given by a strictly increasing pure jump random process with completely monotone Lévy density $k(x)$, satisfying (56). The gamma process of the section 4.2 is the special case when $k(x) = e^{-x}/x$ as defined in equation (47), and this is a completely monotone density. As in the gamma process case, we suppose the prevailing price sell orders $V(t)$ are given by an independent copy of the same Lévy process. The log price process of equation (27) now has the form

$$\ln(p(t)/p(0)) = \alpha_1 U(t) - \alpha_2 V(t) \tag{57}$$

with $\alpha_1 = (\delta + a)^{-1}, \alpha_2 = (\eta + b)^{-1}$. It is shown in the appendix, following the analysis of Leblanc (1997), Knight (1981), Kotani and Watanabe (1982), for the special case of $\alpha_1 = \alpha_2 = \alpha$, that the Lévy measure of the time change is $\widetilde{k}(y) = k_3(\alpha y)$, where k_3 is given by

$$k_3(y) = \int_0^\infty \rho(da) 2ae^{-a^2 y}. \tag{58}$$

The time change Lévy density given explicitly by equation (58) generalizes to the completely monotone class of densities the result we demonstrated earlier for the exponential Lévy measure. We observe interestingly that the time change aggregates the counting of the arrival of orders with the same weighting function to provide the time change. An order by an individual with a mean order arrival rate of $(1/a)$ of size y amounts to $a\exp(-a(a-1)y)$ ticks of the stochastic clock.

The literature relating price changes to economic activity (Tauchen and Pitts (1983), Karpoff (1987), Gallant, Rossi and Tauchen (1992), and Jones, Kaul and Lipson (1994)) has focused on volume and the number of trades as the relevant measure of economic activity. The analysis of this section indicates that the time measure incorporates both the number and size of orders. For $a < 1$, the time or activity measure is positively and exponentially related to the size of orders.

We now consider two important subclasses of the class of completely monotone Lévy densities, and these are the class of generalized gamma convolution densities, and the stable processes with index $\alpha < 1$, and show in each case that the time change lies in the same subclass as well.

4.4 Demand and Supply Shocks as Generalized Gamma Convolutions

An important subclass of completely monotone Lévy measures is the class of generalized gamma convolutions (Bondesson (1992)). A Lévy measure $k(x)$ is in the generalized gamma convolution class if the size weighted Lévy density $xk(x)$, is completely monotone. When we weight the arrival rate or probability by size, we get the expected impact on the process and this hypothesis requires that large jumps have a sufficiently small arrival rate so that their impact is below the impact of smaller jump sizes. Under this hypothesis, small moves dominate large moves in impact and in this section we ask that if the arrival of orders has this property of completely monotone impact, then is this property inherited by the time change. Do large clock ticks also have a lower expected impact on the passage of time than the smaller clock ticks? We show here that indeed this is the case.

For Lévy densities in the class of generalized gamma convolutions, the process is a mixture of gamma processes with the characteristic function for the cumulated prevailing price buy orders $U(t)$ now having the explicit form

$$\phi_{U(t)}(u) = \exp\left(iatu + t\int \log\left(\frac{c}{c - iu}\right) K(dc)\right) \tag{59}$$

where $K(dc)$ is a non-negative measure on the positive half line.[14]

When K is the Dirac delta measure at $c = 1$, $U(t)$ is the standard gamma process $\gamma(t)$ discussed in section 4.2. The generalized gamma convolution class is very wide and includes as special cases the stable processes and the inverse Gaussian processes. It is an infinite parameter class, effectively being parameterized by the measures $K(dc)$. Clearly, the gamma process is the fundamental building block for processes in this class.

Suppose now that $U(t)$ and $V(t)$ are two independent copies of a generalized gamma convolution class of processes with drift a and mixing measure K. Consider the general symmetric price process of equation (57), and evaluate the characteristic function of the log price relative as

$$\begin{aligned} \phi_{\ln(p(t)/p(0))}(u) &= \exp\left(t\int\begin{pmatrix}\log\left(\frac{c}{c-iu\alpha}\right)- \\ \log\left(\frac{c}{c+iu\alpha}\right)\end{pmatrix}K(dc)\right) \\ &= \exp\left(t\int\log\left(\frac{c^2}{c^2+u^2}\right)K(dc)\right) \end{aligned} \tag{60}$$

We wish to determine whether this process can be written as Brownian motion evaluated at a generalized gamma convolution. Suppose for this purpose that $T(t)$ is an increasing process that is a generalized gamma convolution time change with zero drift and mixing measure $\widetilde{K}$. The characteristic function for $T(t)$ has the form

$$\phi_{T(t)}(u) = \exp\left(t\int\log\left(\frac{c}{c-iu}\right)\widetilde{K}(dc)\right) \tag{61}$$

Consider now the characteristic function of Brownian motion evaluated at this generalized gamma convolution time, $W(T(t))$. This is simply obtained by evaluating (61) at $-u^2/(2i)$ and is

$$\phi_{W(T(t))}(u) = \exp\left(t\int\log\left(\frac{\widetilde{c}}{\widetilde{c}+u^2/2}\right)\widetilde{K}(d\widetilde{c})\right). \tag{62}$$

[14]It is required that $K(dc)$ satisfy the integrability conditions

$$\int_0^1 \mid \log(c) \mid K(dc) < \infty \text{ and } \int_1^\infty \frac{1}{c}K(dc) < \infty.$$

The two characteristic functions (60) and (62) are equated on making the change of variable $2\widetilde{c} = c^2$ in (62) and defining

$$\widetilde{K}(d\widetilde{c}) = \frac{1}{c}K(dc). \tag{63}$$

Specifically we have, in the case $K(dc) = g(c)dc$, that the characteristic function for the time change is

$$\phi_{T(t)}(u) = \exp\left(t\int \log\left(\frac{c}{c-iu}\right)\frac{g(\sqrt{2c})}{\sqrt{2c}}dc\right) \tag{64}$$

Equation (64) provides the exact representation of the time change as a generalized gamma convolution.[15]

The weighting that was given to the gamma process with mean $(1/\sqrt{2c})$ in the prevailing price order arrival process is given in the time change to the gamma process with mean $(1/c)$ after multiplication by $(1/\sqrt{2c})$. We now present a useful generalization of the variance gamma model within the generalized gamma convolution family of processes.

4.4.1 Generalizing the Variance Gamma Model to Control Arrival Rates of Different Sizes and Signs

A potentially useful generalization of the variance gamma model studied in Madan and Chang (1998), within the class of generalized gamma convolutions is given by the Lévy measure, $k_{CGYM}(x)$ [16], for an increasing process (the log

[15] In this case the function g must meet the following four integrability conditions:

$$\int_0^1 |\log(c)|\, g(c)dc < \infty; \quad \int_0^1 |\log(c)|\, \frac{g(\sqrt{2c})}{\sqrt{2c}}dc < \infty$$

$$\int_1^\infty \frac{1}{c}g(c)dc < \infty; \quad \int_1^\infty \frac{1}{c}\frac{g(\sqrt{2c})}{\sqrt{2c}}dc < \infty.$$

[16] We name this Lévy measure after Peter Carr and ourselves, in recognition of Peter Carr's request to develop an extension of the variance gamma model in this direction.

price process being obtained by differencing two such processes with possibly different parameters) where

$$k_{CGYM}(x) = \frac{\exp(-Mx)}{Cx^{Y+1}(1+x)^G}. \tag{65}$$

The special case $G = 0, Y = 0$, gives the symmetric variance gamma process when we difference two identical but independent copies of this process. When we take just $G = 0$, we get the Lévy measure studied by Vershik and Yor (1995), for the purpose of combining the gamma and stable laws. We note that the setting $Y = 1$ constitutes the boundary between processes of finite and infinite variation. Further, from the perspective of financial modeling, this Lévy measure, unlike the variance gamma model, allows one to control for the behavior near zero and at infinity separately via the parameters Y and G respectively, and this too separately, for the positive and negative moves, by differentiating these parameters on the two sides. We have here a fairly robust eight parameter stochastic process. It is amenable to statistical work as the characteristic function is available in closed form in terms of the special functions of mathematics. It is shown in the appendix that for the increasing process $X_{CGYM}(t)$,

$$E[\exp(iuX_{CGYM}(t)] = \exp\left(\begin{array}{c} t\frac{\Gamma(1-Y)}{CY}\left[M^{Y+G}U(G, Y+G+1, M)\right. \\ \left. -(M-iu)^{Y+G}U(G, Y+G+1, M-iu)\right] \end{array} \right) \tag{66}$$

where $U(a, b, z) = z^{-a}\ {}_2F_0(a, 1+a-b, -1/z)$, and ${}_2F_0$ is one of the confluent hypergeometric functions.

4.5 Stable Processes as time-changed Brownian motion

The class of increasing stable processes of index $\alpha < 1$ could also be considered as a candidate model for the prevailing price buy and sell order processes. The difference of two independent stable processes of index $\alpha < 1$ is also a time-changed Brownian motion and we enquire into the nature of the time change. For an increasing stable process of index α the Lévy measure is

$$\nu(dx) = \frac{1}{x^{\alpha+1}}dx \text{ for } x > 0, \tag{67}$$

and the difference, $X(t)$, of two independent copies of such a process is the symmetric stable process of index α with characteristic function

$$E[\exp(iuX(t))] = \exp(-tc|u|^{\alpha}), \tag{68}$$

for a positive constant c. We derive from the characteristic functions that the time change is not an increasing stable $\alpha-$process.

The characteristic function of an independent Brownian motion evaluated at an independent increasing stable process of index $\alpha, T(t)$ is given by

$$\begin{aligned} E[\exp(iuW(T(t)))] &= E[\exp(-u^2 T(t)/2)] \\ &= \exp(-t(c/2)|u|^{2\alpha}), \end{aligned}$$

or a symmetric stable process of index 2α.

It follows from this observation that the difference of two increasing stable α processes for $\alpha < 1$, is Brownian motion evaluated at an increasing stable $\alpha/2$ process.

4.6 Scale and Speed Adjusted Time Changes

We now consider, within the class of completely monotone Lévy densities when the time change Lévy process is simply related to the original order arrival process, in that it may involve speeding up or slowing down accompanied by a scaling of the original process. Specifically, we ask when $\gamma_3(t)$ has the form $a\gamma_1(bt)$ for scaling and speed adjustment coefficients a and b. For example when we relate the time change to the volume of transactions, then we are in the case $b = 1$ and $a = 2$. For the scale and speed adjustment to be valid, one must have

$$\begin{aligned} E\left[\exp(iu\gamma_3(t))\right] &= E\left[\exp(iua\gamma_1(bt))\right] \\ &= \exp\left(bt\int_0^\infty (e^{iuax}-1)k(x)dx\right) \end{aligned} \tag{69}$$

or that

$$\exp\left(t\int_0^\infty (e^{iuy}-1)k_3(y)dy\right) = \exp\left(t\int_0^\infty (e^{iuy}-1)k(\frac{y}{a})\frac{b}{a}dy\right) \tag{70}$$

It follows that we must have

$$k_3(y) = \frac{b}{a} k(\frac{y}{a}). \tag{71}$$

Suppose that $k(y) = \int f(a)e^{-ay}da$ and that $k_3(y) = \int f_3(a)e^{-ay}da$ then (71) implies that

$$\begin{aligned} \int f_3(u)e^{-uy}du &= \frac{b}{a}\int f(u)e^{-uy/a}du \\ &= \int bf(av)e^{-vy}dv \end{aligned}$$

It follows that one must have the equality

$$f_3(u) = bf(au) \tag{72}$$

among the transforms.

But by equation (58) on changing variables and writing $\rho(da) = f(a)da$ we note that we must have $f_3(x) = f(\sqrt{x})$. It follows that for a simple scale and speed adjustment in the time change we must have

$$f(\sqrt{x}) = bf(ax). \tag{73}$$

We know the gamma process has this property, and for the gamma process we have

$$\frac{\exp(-x)}{x} = \int_1^\infty e^{-xu}du.$$

Hence for this case $f(u) = 1_{u\geq 1}$. Equation (73) may be expressed as $f(x) = bf(ax^2)$ and for $f(x) = 1_{x\geq 1}$, this condition is satisfied for $a = b = 1$.

More generally, if we define $\widetilde{f}(x) = f(cx)$, then we must have $\widetilde{f}(x) = f(cx) = bf(ac^2x^2) = b\widetilde{f}(x^2)$, if we choose $a = 1/c$. Hence all possible solutions require $\widetilde{f}(x)$ to be proportional to $\widetilde{f}(x^2)$. A wide class of solutions is given by

$$\widetilde{f}(x) = c_1|\ln(x)|^\alpha 1_{[0,1]}(x) + c_2[\ln(x)]^\alpha 1_{[1,\infty)}(x). \tag{74}$$

We note that the wider class of processes for which a scale and speed adjustment is valid contains the gamma Lévy measure and somewhat more generally, the solutions of (74) associated with α, $0 < \alpha < 1$ for $c_1 = 0, c_2 =$

1.[17] An interesting special case occurs when we consider the case $c_1 = 0, c_2 = 1$ and $\alpha = 1$ in (74). This yields[18]

$$k(y) = \int_1^\infty \ln(\xi) e^{-\xi y} d\xi \tag{75}$$

$$= \frac{1}{y} \int_y^\infty \frac{e^{-u}}{u} du. \tag{76}$$

Equation (76) represents the tail of the Lévy measure of the gamma process divided by the lower limit of integration. This is an interesting operation on Lévy measures. In fact if $\rho(dx)$ is a Lévy measure, then defining

$$\widetilde{\rho}(dx) = \frac{dx}{x} \int_x^\infty \rho(dy) \tag{77}$$

[17] We must have $c_1 = 0$ for even if $\alpha = 0$, and $c_2 = 0$ we have

$$k(y) = \int_0^1 e^{-\xi y} d\xi = \frac{1 - e^{-y}}{y}$$

but then $\int_0^\infty (1 \wedge y) k(y) dy = \infty$.

[18] Substituting $\xi = x/y$ we have

$$\begin{aligned} k(y) &= \frac{1}{y} \int_y^\infty (\ln(x) - \ln(y)) e^{-x} dx \\ &= \frac{1}{y} \int_y^\infty \int_y^x \frac{du}{u} e^{-x} dx \\ &= \frac{1}{y} \int_y^\infty \int_u^\infty \frac{e^{-x}}{u} dx du \\ &= \frac{1}{y} \int_y^\infty \frac{e^{-u}}{u} du \end{aligned}$$

we also obtain a Lévy measure since $\int_0^\infty (1 \wedge x)\widetilde{\rho}(dx) < \infty$ holds.[19] For many Lévy measures, and certainly for the gamma process, equation (77) gives new Lévy measures. We also observe that applying (77) twice amounts to taking $c_1 = 0, c_2 = 1$, and $\alpha = 2$ in (74).[20]

---

[19] To verify this we observe

$$\begin{aligned}
\int_0^\infty (1 \wedge x)\widetilde{\rho}(dx) &= \int_0^1 dx \int_x^\infty \rho(dy) + \int_1^\infty \frac{1}{x} \int_x^\infty \rho(dy)dx \\
&= \int_0^\infty (1 \wedge u)\rho(du) + \int_1^\infty \int_1^u \frac{dx}{x}\rho(du) \\
&= \int_0^\infty (1 \wedge u)\rho(du) + \int_1^\infty \ln(u)\rho(du)
\end{aligned}$$

The first integral on the right hand side is finite as ρ is a Lévy measure, and the second is finite provided ρ is sufficiently damped at infinity.

[20] To see this consider

$$k(y) = \int_1^\infty \ln(\xi)^2 e^{-\xi y} d\xi.$$

We may again set $\xi = x/y$ and write that

$$\begin{aligned}
k(y) &= \frac{1}{y}\int_y^\infty (\ln(x) - \ln(y))^2 e^{-x} dx \\
&= \frac{1}{y}\int_y^\infty \int_y^x 2\frac{du}{u} \int_y^v \frac{dv}{v} e^{-x} dx \\
&= \frac{1}{y}\int_y^\infty 2\frac{dv}{v} \int_v^\infty \frac{du}{u} \int_u^\infty e^{-x} dx \\
&= \frac{2}{y}\int_y^\infty \frac{dv}{v} \int_v^\infty \frac{e^{-u}}{u} du
\end{aligned}$$

which is the application of (77) twice to the gamma Lévy measure.

4.7 Brownian Excursions and Equilibrium

We now consider another and equivalent representation of the asset price process that is valid for price processes with completely monotone Lévy densities, a consequence of Krein's theory (Kotani and Watanabe (1982)). In this equivalent formulation, excess demand is modeled as a Brownian motion and price changes are related to the excess demand by a possibly nonlinear response function that we term the *force function*. We show by examples that simple force functions are associated with fairly complex Lévy densities, while the force function for the gamma process is at this writing, to our knowledge, still unknown. Hence the two approaches complement each other and provide a wider class of interesting models than would be possible if we restricted attention to just one of these two equivalent formulations.

Unlike the examples of the earlier sections, the time change and the Brownian motion are no longer independent in the examples of this section. Here the time change represents cumulated volatility, where the latter depends on the Brownian motion itself.

We view the Brownian motion $W(t)$ as a measure of market departure from equilibrium, and view positive values of $W(t)$ as an excess demand in the market driving prices up, while negative values of $W(t)$ are indicative of excess supply driving prices downward. The extent of the price response to the disequilibrium is given by the force function $f(x)$. The price process is

$$S(u) = \int_0^u f(W(s))ds. \tag{78}$$

The process $S(u)$ is a continuous process that increases when Brownian motion is positive at rate $f(W(s))ds$ and decreases when $W(s)$ is negative. Hence if a trader takes a long position during a positive excursion of Brownian motion and successfully reverses his position before a return to zero, there is a pure arbitrage profit to be made. The same holds true for short positions in negative excursions of Brownian motion.

We avoid arbitrage opportunities by restricting trading to occur in equilibrium, i.e. on the zero set of Brownian motion. The period of the Brownian excursion is then akin to a tatonnement period during which the discovery of the equilibrium price takes place. The time domain of equation (78) includes disequilibrium situations when $W(s) \neq 0$ and the market is in the phase of equilibration. We now restrict the price process to equilibrium states when

Brownian motion is zero. To perform this restriction to the equilibrium domain, we define $\sigma(t)$ to be the inverse local time of Brownian motion at zero. Our price process is the bounded variation process (see the integrability condition (80) below)

$$\ln(p(t)/p(0)) = \int_0^{\sigma(t)} ds f(W(s)) \tag{79}$$

This is a pure jump process as it is $S(\sigma(t))$ and it inherits the jump properties from the inverse local time.

Inverse local time is a Lévy process with Lévy measure $L(x)$,

$$L(x) = \frac{1}{\sqrt{2\pi} x^{3/2}},\ x > 0$$

Hence, this is a high activity jump process that jumps at a zero of Brownian motion to another zero of Brownian motion, with the size of the jump being the length of the Brownian excursion. Each jump time of inverse local time is an equilibrium trading time and there are infinitely many trading opportunities in any interval of time.

The price process (79) includes Lévy processes of infinite variation but finite quadratic variation. To obtain processes of bounded variation one must restrict the class of force functions to those that meet the integrability condition

$$\int_{-K}^{K} dx \mid f(x) \mid < \infty \tag{80}$$

for all K. Under this integrability condition (that essentially prevents an infinite force at zero), the two components of positive and negative moves may be constructed as separate processes. One may write the price process as

$$\ln(p(t)/p(0)) = \int_0^{\sigma(t)} ds f^+(W(s)) - \int_0^{\sigma(t)} ds f^-(W(s)) \tag{81}$$

where $f^+(x) = f(x)\mathbf{1}_{(x \geqq 0)}$; $f^-(x) = f(x)\mathbf{1}_{(x \leqq 0)}$. [21] We now model the term

[21] It is interesting to note that when the integrability condition is not met and there is an infinite activity near the origin that only sums up when we square the moves, one

$(\delta + a)^{-1}U(t)$ of equation (27) by $\int_0^{\sigma(t)} dsf^+(W(s))$ while $(\eta + b)^{-1}V(t)$ is given by $\int_0^{\sigma(t)} dsf^-(W(s))$.

It is interesting to note that during an excursion of the Brownian motion away from 0, the market is equilibrating and the clock $\sigma(t)$ stops and does not count time. The only times we count and the only firm trading prices we permit are those when we have equilibrium, and $W(\sigma(t)) = 0$. We measure time by time spent in equilibrium and we note that $\sigma(t)$ is strictly increasing in t.

Some interesting facts about the Lévy measure for the price process may be inferred directly from the structure of the force function. For example, one may show that if the force function is zero in an interval around zero, then only excursions of a certain significant departure from zero contribute to a price movement, and one may infer that the arrival rate of the process is finite.[22] Similarly one may show that if the force function is strictly positive in absolute value in an open interval around zero then the arrival rate of the Lévy measure is infinite. Hence, we can conclude that economies that are dormant around the equilibrium point in that they have no force, have finite arrival rate Lévy measures. Furthermore, if the force function of one economy dominates the force function of another, then the tail of the Lévy measure of the economy with the dominating force function is heavier. Greater force generally means greater activity in terms of the arrival rate of price changes.

The price process of equation (79) may once again be expressed as time-changed Brownian motion. Define $\psi(y)$ by $\psi'(y) = f(y)$ and $\psi(0) = 0$, let

$$F(x) = \int_0^x \psi(y)dy. \tag{82}$$

may approximate the process by considering the difference of increasing processes that truncate the force function at values of x below ε in absolute value and obtain the infinite variation process as the limit as we let ε tend to zero, provided as a sufficient condition that $\int_0 x^\alpha f(x)dx < \infty$ for some value $\alpha < 1/2$. For the cases we consider this condition is met. In this sense our decomposition into the difference of two increasing processes is quite fundamental.

[22] This property can be formally proved by integrating the force functions with respect to the Ito measure for the height m of the excursion (see Yor (1995) page 68) which is $(1/m^2)dm$.

Ito's lemma may now be applied to $F(W(t))$ in the following way:

$$0 = F(W(\sigma(t))) = \int_0^{\sigma(t)} \psi(W(s))dW(s) + \frac{1}{2}\int_0^{\sigma(t)} f(W(s))ds \qquad (83)$$

It follows from equation (79) substituted in (83) that

$$\ln(p(t)/p(0)) = -2\int_0^{\sigma(t)} \psi(W(s))dW(s) \qquad (84)$$

Hence the probability law of $\ln(p(t)/p(0))$ is that of Brownian motion evaluated at $4\int_0^{\sigma(t)} \psi^2(W(s))ds$. However, here the Brownian motion and the time change are not independent processes. The time change is in fact constructed from the original Brownian motion as [23]

$$T(t) = 4\int_0^{\sigma(t)} \psi(W(s))^2 ds.$$

We note that the time change cumulates the instantaneous volatility of the price process, and observe that this volatility depends on the Brownian motion.

An interesting special case of such equilibrium processes derived from Brownian excursions is obtained by considering the case of

$$f(x) = a(x^+)^m + b(x^-)^m \qquad (85)$$

For $a > 0$ and $b = -a$, the log price process (79) is a symmetric stable process of index $\alpha = (m + 1/2)^{-1}$. This process can be written as Brownian motion evaluated at a dependent time constructed from the price process using (84) as the quadratic variation of the price process. The Brownian motion

[23] It is possible for a continuous martingale to be written as a Brownian motion evaluated at an independent time change given by its quadratic characteristic. However, as shown by Ocone (1993), Dubins, Émery, Yor (1993), a necessary and sufficient condition for this to be possible is that the probability law of stochastic integrals with respect to the martingale coincide for all integrands that have absolute value unity. For the force functions f, and the associated function ψ, this is unlikely to be the case.

may also be constructed out of the price process, as the price process evaluated at the inverse of the quadratic variation. We have here an alternative representation of stable α, $\alpha < 1$, processes as time-changed Brownian motions (as opposed to the representation of section (4.5)), that do not invoke independence between the Brownian and the time change.

It is instructive to consider the relationship between the Lévy measures of the stable processes and the force functions of the representations (79). We observe that when m is large, the force is high at large values of x and the Lévy measure is accordingly lower at these high values of x. Similarly, for low values of x, the force is higher for low values of m, and once again these low values of m have the lower Lévy measure. It therefore appears that tails for the Lévy measure that are less heavy than the stable laws would require a force function that dominates the polynomials.

4.8 Characteristic Functions for Price Processes Based on Brownian Excursions

For the econometric evaluation and identification of price processes from data on financial prices, the characteristic function of the log price relative is a very useful and fundamental construct. For all the cases considered in sections 4.1 through 4.6, we have explicit characteristic functions for the log price relative. In this section we present an algorithm to evaluate the characteristic function for the Brownian excursion model presented in section 4.7, based on the results of Revuz and Yor (1994).

We are interested in the characteristic function

$$\phi_{\ln(p(t)/p(0))}(u) = E\left[\exp\left(iu\int_0^{\sigma(t)} f(W(s))ds\right)\right], \tag{86}$$

where $\sigma(t)$ is the inverse local time of the Brownian motion $W(t)$. Writing the function $f = f^{+} - f^{-}$, noting that f is positive for positive arguments and negative for negative arguments, we may rewrite (86) as

$$\phi_{\ln(p(t)/p(0))}(u) = E\left[\exp\left(iu\int_0^{\sigma(t)} f^{+}(W(s))ds - iu\int_0^{\sigma(t)} f^{-}(W(s))ds\right)\right]. \tag{87}$$

By the Ray-Knight theorem, the two components are independent and the result follows on multiplication of the two expectations. Hence,

$$\phi_{\ln(p(t)/p(0))}(u) = E\left[\exp\left(iu\int_0^{\sigma(t)} f^+(W(s))ds\right)\right] \times E\left[\exp\left(-iu\int_0^{\sigma(t)} f^-(W(s))ds\right)\right]. \quad (88)$$

The result follows on evaluating each of the expectations. It is shown in Revuz and Yor (1994), that the Laplace transform is given by

$$E\left[\exp\left(-\lambda\int_0^{\sigma(t)} f^+(W(s))ds\right)\right] = \exp(t\psi'(0^+)/2) \quad (89)$$

where $\psi(x)$ is the unique positive, decreasing solution to the Sturm-Liouville equation

$$\frac{1}{2}\psi''(x) = \lambda f^+(x)\psi(x)$$

subject to the boundary conditions $\psi(0) = 1$, $\psi(\infty) = 0$, when the function f meets the condition $\int_0^\infty xf^+(x)dx = \infty$. Here we present an analysis based on the Ray-Knight theorem that uses methods more commonly employed in the economics literature, and initiated by Cox, Ingersoll and Ross (1985).

By the Ray-Knight theorem one may write the process for the price increases as

$$\int_0^{\sigma(t)} f^+(W(s))ds = \int_0^\infty dxf^+(x)L^x_{\sigma(t)}(W), \quad (90)$$

where $L^x_{\sigma(t)}(W) = Z(x)$ is the local time of the Brownian motion W at x between 0 and $\sigma(t)$, the inverse local time at zero of W. The process $Z(x)$ viewed as a process in the space variable, for fixed t, is a Feller diffusion and is in fact a squared Bessel process of dimension 0, starting at t, or in finance terms a CIR process (see Geman and Yor 1993), that satisfies the stochastic differential equation

$$dZ(x) = 2\sqrt{Z(x)}dB(x); \quad Z(0) = t \quad (91)$$

for a standard Brownian motion $B(x)$ in the space variable. We are therefore equivalently interested in the Laplace transform of $\int_0^\infty dx f^+(x)Z(x)$. We proceed by considering this problem in the familiar way of analyzing term structure models and define

$$G(y,Z) = E\left[\exp\left(-\lambda\int_y^\infty f^+(x)Z(x)dx\right) \mid Z(y) = Z\right]. \tag{92}$$

The partial differential equation for G may be derived as

$$G_y + 2ZG_{ZZ} - \lambda f(y)Z = 0, \tag{93}$$

which must be solved for the boundary conditions $G(y,0) = 1$, and $G(\infty, Z) = 0$ [24] . It is well known (see eg. Revuz and Yor (1994) page 424 Theorem 4.7) that the solution in Z is of the form

$$G(y,Z) = \exp(b(y)Z) \tag{94}$$

where the function b satisfies the Ricatti differential equation (also classically obtained in finance for CIR type models of interest rates or stochastic volatility)

$$b' + 2b^2 = \lambda f^+.$$

The Sturm-Liouville equation follows on making the substitution $b = \frac{1}{2}\frac{\psi'}{\psi}$, and the result follows on evaluating $G(0, Z(0)) = G(0,t)$. We now consider some explicit examples where the characteristic function of the log price relative may be explicitly evaluated, beyond the polynomial case for the force function f for which we have already observed that we obtain a stable law for the price process.

We consider two further cases, one with a decreasing force function, and the other with an exponentially increasing force function. The second example has a force function that dominates the polynomials yielding the stable laws that we have already reported on.

[24]The condition at infinity is related to the requirement that $f^+(x)x$ integrates to infinity. When this is finite, the condition at infinity is different, but we shall only be concerned with cases where this condition is met.

4.8.1 Example with Diminishing Force Function

Suppose that

$$f^+(x) = \frac{1}{(kx+l)^2}.$$

Let Z_x be the Feller diffusion associated with the local times at x evaluated at the inverse local time at t. As noted earlier, $dZ_x = 2\sqrt{Z_x}dB(x)$ for a standard Brownian motion B, and $Z_0 = t$. Define $\varphi(x) = (kx+l)^{-1}$ and consider $Y_x = \varphi(x)Z_x$. By construction $Y_0 = t/l$, and an application of Ito's lemma shows that

$$dY_x = -\frac{k}{(kx+l)^2}Z_x dx + 2\varphi(x)\sqrt{Z_x}dB(x).$$

Writing the martingale $\int_0^x 2\varphi(y)\sqrt{Z_y}dB(y)$ in its Dubins-Schwarz form: $w(A_x)$, where $(w(u), u \geq 0)$ is a Brownian motion, and $A_x = 4\int_0^x dy\varphi^2(y)Z_y$, we may represent $\{Y_x\}$ as: $\gamma(A_x)$, where

$$\gamma(t) = \frac{t}{l} - \frac{k}{4}t + w(t).$$

It follows that $\gamma\left(4\int_0^\infty dy\,(ky+l)^{-2}\,Z_y\right)$ is Y_∞, which by construction is zero. Hence,

$$4\int_0^{\sigma(t)} f^+(W(s))ds = 4\int_0^\infty dx\,(kx+l)^{-2}\,Z_x = T_0(\gamma)$$

the time at which the Brownian motion γ with drift $-k/4$ and initial value t/l reaches zero. This is the same as the first passage time distribution of Brownian motion starting at zero and with drift $k/4$ reaching t/l. The density of $T_0(\gamma)$ is given by

$$P(T_0(\gamma)\epsilon ds) = \frac{t/l}{\sqrt{2\pi s^3}}\exp\left(-\frac{(t/l)^2}{2s}\right)\exp\left(\frac{tk}{4l} - \frac{k^2 s}{32}\right)ds.$$

The Laplace transform is given by

$$E\left[\exp\left(-\lambda T_0(\gamma)\right)\right] = \exp\left(-\frac{t}{l}\left(\sqrt{\frac{k^2}{16}+2\lambda}-\frac{k}{4}\right)\right)$$

and the Lévy measure is

$$k(x) = \frac{1}{l}\frac{\exp\left(-\frac{k^2}{32}x\right)}{\sqrt{2\pi x^3}}.$$

We observe in this example that the force function and the Lévy measure are both inversely related to the parameters k and l, and hence are positively related to each other.

4.8.2 Example with Exponentially Increasing Force Function

Suppose that

$$f^+(x) = \theta\exp(\alpha x), \quad \alpha,\theta > 0 \tag{95}$$

For this case, following Jeanblanc, Pitman and Yor (1997), Example 6, we observe that the solution to the Sturm Liouville equation with the boundary condition $\psi(0) = 1$ and $\psi(\infty) = 0$, is

$$\psi(x) = \frac{K_0\left(\frac{2\sqrt{2\lambda\theta}}{\alpha}\exp\left(\frac{\alpha}{2}x\right)\right)}{K_0\left(\frac{2\sqrt{2\lambda\theta}}{\alpha}\right)}$$

where K_0 is the modified Bessel function of second kind of order zero. It follows on differentiation, noting that $K_1 = -K_0^{'}$ and substituting into (89) that

$$E\left[\exp\left(-\lambda\int\limits_0^{\sigma(t)} f^+(W(s))ds\right)\right] = \exp\left(-\frac{t}{2}\frac{\sqrt{2\lambda\theta}K_1\left(\frac{2\sqrt{2\lambda\theta}}{\alpha}\right)}{K_0\left(\frac{2\sqrt{2\lambda\theta}}{\alpha}\right)}\right). \tag{96}$$

For the Lévy measure associated with this Laplace transform we note from

Donati-Martin and Yor (1997) page 1055 that

$$\sqrt{\xi}\frac{K_1(\sqrt{\xi})}{K_0(\sqrt{\xi})} = \xi \int_0^\infty \exp\left(-\xi y\right) H_{-1}(y)dy = -\int_0^\infty (1 - \exp(-\xi y) \frac{\partial}{\partial y} H_{-1}(y)dy$$

Substituting $2\sqrt{2\lambda\theta}/\alpha$ for $\sqrt{\xi}$ and making the change of variable $8\theta y/\alpha^2 = z$ we obtain that the Lévy measure for this process is

$$k(x) = \frac{\alpha^3}{16\theta}\frac{1}{\pi^2}\int_0^\infty dz \exp\left(-z\frac{\alpha^2 x}{8\theta}\right)\frac{1}{J_0^2(\sqrt{z}) + Y_0^2(\sqrt{z})}$$

where we have differentiated H_{-1} using the definition provided in Donati-Martin and Yor (1997) equation (6.6).

5 Time Changes related to Demand and Supply Shocks in the Limit

Recently, Barndorff-Nielsen (1997) has proposed the normal inverse Gaussian distribution as a possible model for the stock price process. This process may also be represented as a time-changed Brownian motion, where the time change $T(t)$ is the first passage time of another independent Brownian motion with drift to the level t. The time change is therefore an inverse Gaussian process and as one evaluates a Brownian motion at this time, this suggests the nomenclature normal inverse Gaussian. An interesting question from the perspective of this paper is whether such a process may be represented as the difference of two increasing processes that constitute the price responses to demand and supply shocks, respectively. However, this is not possible as the normal inverse Gaussian process is one of infinite variation.

We note that the inverse Gaussian process is a homogeneous Lévy process that is in fact a stable process of index $\alpha = 1/2$. We observed in section (4.5), that if $2\alpha < 1$, then time changing Brownian motion with such a process leads to the symmetric stable process of index $\alpha < 1$. For $\alpha = 1/2$, we observe below that the process is of infinite variation.

In general, for $W(T(t))$ to be a process of bounded variation we must have that

$$\int (1 \wedge |x|)\widetilde{\nu}(dx) < \infty, \tag{97}$$

where $\widetilde{\nu}$ is the Lévy measure of the time-changed Brownian motion. We may relate $\widetilde{\nu}$ to the Lévy measure ν, of the time change by

$$\widetilde{\nu}(dx) = dx \int \nu(dy) \frac{\exp(-\frac{x^2}{2y})}{\sqrt{2\pi y}}. \tag{98}$$

The time changed process is of bounded variation just if (see the Appendix for a proof),

$$\int_0 \nu(dy)\sqrt{y} < \infty. \tag{99}$$

For the inverse Gaussian time change, we see from (67) for $\alpha = 1/2$ that (99) is infinite. It follows that the normal inverse Gaussian process is a time-changed Brownian motion of infinite variation and therefore it cannot be expressed as the difference of two increasing processes. It may, however, be approximated by such processes by ignoring jumps of absolute size below ε and then letting ε tend to zero.

From the perspective of Brownian excursions, we offer the example of a force function that diverges to infinity at 0, and results in an infinite variation Lévy process. The specific force function is studied in Donati-Martin and Yor (1997) and is given by

$$f(x) = sign(x)\frac{1}{\exp(2\theta \mid x \mid) - 1}.$$

It is shown in Donati-Martin and Yor (1997), that the Laplace transform of the positive moves

$$E\left[\exp\left(-\lambda \int_0^{\sigma(t)} f^+(W(s))ds\right)\right] = \exp\left(-\frac{t\lambda}{2\theta}\left\{\begin{array}{c} -2\gamma - \Psi\left(1 + i\sqrt{\frac{\lambda}{2}}\right) \\ -\Psi\left(1 - i\sqrt{\frac{\lambda}{2}}\right) \end{array}\right\}\right) \tag{100}$$

where γ is Euler's constant and $\Psi(x) = \Gamma'(x)/\Gamma(x)$. It is shown in the Appendix that the characteristic function of the log price process is given by

$$\phi_{\ln(p(t)/p(0))}(u) = \exp\left(-\frac{t}{\theta}\sum_{n=1}^{\infty}\frac{4u^2 n}{u^2 + 4n}\right), \tag{101}$$

and the Lévy measure is

$$k(x) = \frac{\partial^2}{\partial x^2}\frac{\exp(-4x)}{1 - \exp(-4x)}. \tag{102}$$

6 Conclusion

We argue in this paper that price processes, representing market responses to underlying uncertainties given by the increasing random processes of cumulated demand and supply shocks are bounded variation semimartingales. Furthermore, being semimartingales (an implication of the no arbitrage condition) they are time-changed Brownian motions-as shown by Monroe (1978). The focus of attention in modeling the price process then shifts to modeling the time change. We show, by various examples, that one may generally relate this time change to a measure of economic activity and hence deduce that it is an increasing process with local uncertainty. This has the implication that the time change is a pure jump process and hence, so is the price process. Specifically, we conclude that the process for the price of a traded asset should not be modeled as possessing a continuous martingale or diffusion component. This result is contrary to the fundamental paradigm of modern pricing theory and the dominant practice of the past twenty five years. So far our investigation has concentrated on homogeneous time changes that are independent of the price path, and we recognize that it may be necessary to develop methodologies that allow for time inhomogeneous and price path dependent stochastic time changes. We anticipate that future research will address these issues.

The specific time changes considered in this paper are compound Poisson processes, gamma processes, general Lévy processes with completely monotone Lévy densities, generalized gamma convolutions and the inverse local time of Brownian motion at zero. In each case we exhibit the price process as a finite variation process that is the difference of two increasing processes, one recording the price increases and the other the price decreases. In each case we show how the price process may be viewed as Brownian motion evaluated at a random time that is related to the sum of the processes being differenced to get the price process. We interpret the time change in all cases as a measure of economic activity. For the widest class of processes considered, we show that time is to be measured as a size weighted cumulation of orders. In this sense, the correct analytical measure turns out to be a combination of the number of trades and volume proxies often considered in the empirical literature. In addition we provide a wide class of operational models, with associated test procedures, for the price processes of market economies in continuous time.

References

[1] Abramowitz, M. and I. A. Stegun (1972), *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series, 55, Washington, D.C.

[2] Ané, T. and H. Geman (1997), "Order Flow, Transactions Clock and Normality of Asset Returns," *Working Paper, University of Paris IX Dauphine.*

[3] Bakshi, G., Cao C. and Z. Chen (1997), "Empirical Performance of Alternative Option Pricing Models," forthcoming *Journal of Finance.*

[4] Bakshi, G. and D. B. Madan (1997), "A Simplified Approach to the Valuation of Options," *Working Paper, College of Business and Management, University of Maryland*, College Park, MD. 20742.

[5] Bates, D. (1996), "Post-'87 Crash Fears in S&P500 Futures Options," working paper, *University of Iowa*, Iowa City, Iowa 52242-1000.

[6] Black, F. and M. Scholes (1973), "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637-654.

[7] Bondesson, L. (1992), *Generalized Gamma Convolutions and Related Classes of Distributions and Densities*, Lecture Notes in Statistics 76, Springer-Verlag, Heidelberg.

[8] Clark, P.K. (1973), "A Subordinated Stochastic Process with Finite Variance for Speculative Prices," *Econometrica*, 41, 135-155.

[9] Cox, J.C., S.A. Ross and M. Rubinstein (1979), "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, 7, 229-263.

[10] Cox, J. C. . and C. F. Huang (1989), "Optimum Consumption and Investment Portfolio Policies When Asset Prices Follow a Diffusion Process," *Journal of Economic Theory*, 49, 33-83.

[11] Cox, J., J. Ingersoll, and S. Ross (1985), "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53, 385-408.

[12] Delbaen F. and W. Schachermayer (1994), "A General Version of the Fundamental Theorem of Asset Pricing," *Mathematische Annalen*, 300, 563-520.

[13] Detemple, J. and S. Murthy (1994), "Intertemporal Asset Pricing with Heterogeneous Beliefs," *Journal of Economic Theory*, 62, 2, 294-

[14] Donati-Martin, C. and Marc Yor (1997), "Some Brownian Functionals and Their Laws," *The Annals of Applied Probability*," 25, 1011-1058.

[15] Dubins, L.E., M. Émery and M. Yor (1993), "On the Lévy Transformation of Brownian Motions and Continuous Martingales," *Seminaire de Probabilité, XXVII, Lecture Notes in Mathematics*, 1557, 122-132.

[16] Duffie, D. and C. Huang (1985), "Implementing Arrow-Debreu Equilibria by Continuous Trading of Few Long-lived Securities," *Econometrica*, 53, 1337-1356.

[17] Dumas, B. (1989), "Two-Person Dynamic Equilibrium in the Capital Market," *Review of Financial Studies*, 2, 157-188.

[18] Gallant, A. R., P. E. Rossi, and G. Tauchen (1992), "Stock Prices and Volume," *Review of Financial Studies*, 5, 199-242.

[19] Geman, H. and M. Yor (1993), "Bessel Processes, Asian Options and Perpetuities," *Mathematical Finance*, 349-375.

[20] Hakansson, N.H. (1979), "The fantastic world of finance: progress and the free lunch," *Journal of Financial and Quantitative Analysis*, 14, 717-734.

[21] Harrison, M. and D. Kreps (1979), "Martingales and Arbitrage in Multiperiod Security Markets," *Journal of Economic Theory*, 20, 381-408.

[22] Harrison, M. and S. Pliska (1981), "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stochastic Processes and Their Applications*, 11, 215-260.

[23] He, H. and H. Leland (1993), "On Equilibrium Asset Price Processes," *Review of Financial Studies*, 6, 593-617.

[24] Heston, S. L. (1993), "Invisible Parameters in Option Prices," *Journal of Finance*, 48, 3, 933-947.

[25] Jarrow, R.A. and D.B. Madan (1995), "Option Pricing Using the Term Structure of Interest Rates to Hedge Systematic Discontinuities in Asset Returns," *Mathematical Finance*, 5, 4, 311-336.

[26] Jeanblanc, M. , J. Pitman and M. Yor (1997), "The Feynman-Kac formula and decomposition of Brownian paths," *Computational and Applied Mathematics, Special Issue on Stochastic Analysis*, 16, 1, 27-52.

[27] Jones, E.P. (1984), "Option Arbitrage and Strategy with Large Price Changes," *Journal of Financial Economics*, 13, 91-113.

[28] Jones, C., G. Kaul and M. L. Lipson (1994), "Transactions, Volumes and Volatility, *Review of Financial Studies*, 7, 631-651.

[29] Karlin, S. and H. Taylor (1981), *A Second Course in Stochastic Processes*, Academic Press, London.

[30] Karpoff, Jonathan M. (1987), "The Relation Between Price Changes and Trading Volume: a Survey," *Journal of Financial and Quantitative Analysis*, 22, 109-126.

[31] Knight, F.B. (1981), "Characterization of the Lévy measure of Inverse Local Time for Gap Diffusions," *Seminar on Stochastic Processes,* Birkhauser, 53-78.

[32] Kreps, D. (1981), "Arbitrage and Equilibrium in Economies with Infinitely Many Commodities," *Journal of Mathematical Economics*, 8, 15-35.

[33] Kotani, S. and S. Watanabe (1982), "Krein's Spectral Theory of Strings and Generalized Diffusion Processes," In: *Functional Analysis in Markov Processes*, ed. Fukushima, Lecture Notes in Mathematics 923, Springer-Verlag.

[34] Leblanc, B. (1997), *Modélisation de la Volatilité d'un Actif Financier et Applications*, Dissertation, University de Paris VII-Denis Diderot.

[35] Madan, D.B. and F. Milne (1991), "Option Pricing with V.G. Martingale Components," *Mathematical Finance*, 1, 4, 39-55.

[36] Madan, D.B. and E. Chang (1998), "The Variance Gamma Option Pricing Model," Working Paper, *University of Maryland*, College Park, MD. 20742.

[37] Mandelbrot, B. (1963), "New Methods in Statistical Economics," *Journal of Political Economy*, 61, 421-440.

[38] McCulloch, J. H. (1978), "Continuous Time Processes with Stable Increments," *Journal of Business*, 51, 4, 601-620.

[39] Merton, R.C. (1971), "Optimum Consumption and Portfolio Rules in a Continuous Time Model," *Journal of Economic Theory*, 3, 373-413.

[40] Merton, R.C. (1973), "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4, 141-183.

[41] Merton, R.C. (1976), "Option Pricing When Underlying Stock Returns are Discontinuous," *Journal of Financial Economics*, 3, 125-144.

[42] Monroe, Itrel (1978), "Processes that can be embedded in Brownian Motion," *The Annals of Probability*, 6, 1, 42-56.

[43] Ocone, D.L. (1993), "A Symmetry Characterization of Conditionally Independent Increment Martingales," *Barcelona Seminar on Stochastic Analysis, St. Felice de Guixols, 1991,* D. Nualart and M. Sanz eds., Birkhauser, New York.

[44] Press, J.S. (1967), "A Compound Events Model for Security Prices," *Journal of Business*, 40, 317-335.

[45] Revuz, D. and M. Yor (1994), *Continuous Martingales and Brownian Motion*, Springer-Verlag, Berlin.

[46] Tauchen G. and M. Pitts (1983), "The Price-Volume Relationship on Speculative Markets," *Econometrica*, 51, 485-505.

[47] Vershik, A. and M. Yor (1995), "Multiplicativité du processus gamma et étude asymptotique des lois stables d'indice α, lorsque α tend vers 0," Prépublication N^o 289 du Laboratoire de Probabilités de L'Université Paris VI.

[48] Yor, M. (1995), *Local Times and Excursions for Brownian Motion: A Concise Introduction*, Lecciones en Matemáticas, Número 1, Postgrado de Matemáticas, Facultdad de Ciencias, Universidad Central de Venezuela.

APPENDIX

Derivation of equation (55):
We may write the right hand side of (54) as

$$\begin{aligned} \alpha_1 N_d^2 - \alpha_2 N_s^2 &= (\sqrt{\alpha_1}N_d - \sqrt{\alpha_2}N_s)(\sqrt{\alpha_1}N_d + \sqrt{\alpha_2}N_s) \\ &= \sqrt{\alpha_1+\alpha_2}M(\sqrt{\alpha_1}N_d + \sqrt{\alpha_2}N_s) \end{aligned} \tag{A1}$$

where $M = (\sqrt{\alpha_1}N_d - \sqrt{\alpha_2}N_s)/\sqrt{\alpha_1+\alpha_2}$ is another standard normal variate. Now project $\sqrt{\alpha_1}N_d + \sqrt{\alpha_2}N_s$ on M and write

$$\sqrt{\alpha_1}N_d + \sqrt{\alpha_2}N_s = \frac{\alpha_1-\alpha_2}{\sqrt{\alpha_1+\alpha_2}}M + \frac{2\sqrt{\alpha_1\alpha_2}}{\sqrt{\alpha_1+\alpha_2}}\widetilde{M} \tag{A2}$$

where $\widetilde{M}$ is a standard normal variate independent of M. Substitution of (A2) into (A1) shows that one may write

$$\alpha_1 N_d^2 - \alpha_2 N_s^2 = (\alpha_1-\alpha_2)M^2 + 2\sqrt{\alpha_1\alpha_2}|M|\widetilde{M} \tag{A3}$$

If we define by $\gamma_3(1/2)$, by the relation $M^2 = 2\gamma_3(1/2)$, one may then write (A3) as

$$\alpha_1 N_d^2 - \alpha_2 N_s^2 = (\alpha_1-\alpha_2)2\gamma_3(1/2) + 2\sqrt{\alpha_1\alpha_2}\sqrt{2\gamma_3(1/2)}\widetilde{M} \tag{A4}$$

and division of (A4) by 2, noting (54) yields

$$\ln\left(\frac{p\left(\frac{1}{2\kappa}\right)}{p(0)}\right) = (\alpha_1-\alpha_2)\gamma_3(1/2) + \sqrt{2\alpha_1\alpha_2}\sqrt{\gamma_3(1/2)}\widetilde{M} \tag{A5}$$

or the result that the log price process is Brownian motion with drift $(\alpha_1-\alpha_2)$ and volatility $\sqrt{2\alpha_1\alpha_2}$ evaluated at $\gamma_3(t)$.

We see from (A5) that the price process is basically the difference of squares of normals. The nonnegative prevailing price buy and sell orders are essentially the squares of Gaussian variates , N_d, and N_s, respectively. The time change, M, is the square of the excess demand, $\left(\sqrt{B}-\sqrt{S}\right)^2$, which is the activity measure in this case. ■

Derivation of Equation (58):

First note that the characteristic function of $\ln(p(t)/p(0))$ may be written as

$$\phi_{\ln(p(t)/p(0))}(u) = \exp\left(-2t\int_0^\infty (1-\cos(u\alpha x))k(x)dx\right)$$

On the other hand the characteristic function of $\sigma W(t) = Y(t)$ evaluated at a time change $\gamma_3(t)$ conditional on the time change is

$$\phi_{Y(\gamma_3(t))|\gamma_3(t)}(u) = \exp\left(-\frac{\sigma^2 u^2 \gamma_3(t)}{2}\right)$$

Suppose that $\gamma_3(t)$ is a Lévy process with Lévy measure $k_3(x)dx$, then the characteristic function of $Y(\gamma_3(t))$ is given by

$$\phi_{Y(\gamma_3(t))}(u) = \exp\left(-t\int_0^\infty (1-e^{-\frac{\sigma^2u^2}{2}x})k_3(x)dx\right)$$

For the log price process to be a time-changed Brownian motion, using the Lévy process $\gamma_3(t)$ for the time change, one must have

$$2\int_0^\infty (1-\cos(u\alpha x))k(x)dx = \int_0^\infty (1-e^{-\frac{\sigma^2u^2}{2}x})k_3(x)dx \tag{A6}$$

We may let $\sigma^2/2 = \alpha$, $\widetilde{k}(y) = k(y/\alpha)$, $\widetilde{k}_3(y) = k_3(2y/\sigma^2)$ and then write (A6) as

$$2\int_0^\infty (1-\cos(ux))\widetilde{k}(x)dx = \int_0^\infty (1-e^{-u^2x})\widetilde{k}_3(x)dx \tag{A7}$$

Differentiating (A7) with respect to u yields

$$\int_0^\infty \frac{\sin(ux)}{u}x\widetilde{k}(x)dx = \int_0^\infty e^{-u^2x}x\widetilde{k}_3(x)dx \tag{A8}$$

We now recall that

$$\frac{\sin(ux)}{u} = \frac{1}{2}\int_{-\infty}^\infty 1_{|y|<x}e^{iuy}dy \tag{A9}$$

and

$$e^{-u^2 x} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi x}} e^{-\frac{y^2}{4x}} e^{iuy} dy \tag{A10}$$

Substituting (A10) and (A9) into (A8) and using the uniqueness of Fourier transforms, we deduce that for each y

$$\int_0^{\infty} x\widetilde{k}(x)\frac{1}{2}1_{|y|<x} dx = \int_0^{\infty} x\widetilde{k}_3(x)\frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{x}} e^{-\frac{y^2}{4x}} dx \tag{A11}$$

Differenting (A11) with respect to $y \geq 0$ yields

$$\widetilde{k}(y) = \frac{1}{2}\int_0^{\infty} \frac{\widetilde{k}_3(x)}{\sqrt{\pi x}} e^{-\frac{y^2}{4x}} dx \tag{A12}$$

Equation (A12) may be solved for $\widetilde{k}_3(x)$ satisfying (58) when $\widetilde{k}(y)$ is given as the Laplace equation (56). To observe this we recall that

$$e^{-ay} = \int_0^{\infty} \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{y^2 t}{2}} e^{-\frac{a^2}{2t}} dt \tag{A13}$$

Employing (A13) we may write

$$\begin{aligned} \widetilde{k}(y) &= \int e^{-ay}\rho(da) \\ &= \int \rho(da) \int_0^{\infty} \frac{dt}{\sqrt{2\pi t^3}} a e^{-\frac{y^2 t}{2}} e^{-\frac{a^2}{2t}} \end{aligned}$$

Making the change of variable $t = \frac{1}{2x}$ we obtain

$$\widetilde{k}(y) = \int \rho(da) \int_0^{\infty} \frac{dx}{\sqrt{\pi x}} a e^{-\frac{y^2}{4x}} e^{-a^2 x}$$

It follows that defining $\widetilde{k}_3$ by equation (58) satisfies (A11) as was to be shown. ■

Derivation of equation (66):

We wish to evaluate the integral

$$A = \int_0^\infty \frac{\exp(-Mx)}{Cx^{Y+1}(1+x)^G}(e^{iux}-1)dx. \tag{A14}$$

For this we first consider the general form for $\delta < 1$ (the case of finite variation)

$$\begin{aligned}\int_0^\infty \left(e^{iux}-1\right)\frac{e^{-\gamma x}}{x^{\delta+1}}dx &= -\frac{1}{\delta}x^{-\delta}(e^{-(\gamma-iu)x}-e^{-\gamma x})\,|_0^\infty + \\ & \int_0^\infty \frac{1}{\delta}x^{-\delta}\left(\gamma e^{-\gamma x}-(\gamma-iu)e^{-(\gamma-iu)x}\right)dx \\ &= \frac{\Gamma(1-\delta)}{\delta}\left(\gamma^\delta-(\gamma-iu)^\delta\right). \end{aligned} \tag{A15}$$

Coming now to the integration of (A14) we first note that

$$\frac{1}{(1+x)^G} = \int_0^\infty \frac{1}{\Gamma(G)}y^{G-1}e^{-(1+x)y}dy \tag{A16}$$

Substitution of (A16) into (A14) and reversing the order of integration yields that

$$A = \int_0^\infty \frac{y^{G-1}e^{-y}}{\Gamma(G)C}\int_0^\infty (e^{iux}-1)\frac{e^{-(M+y)x}}{x^{Y+1}}dx \tag{A17}$$

We now employ (A15) in (A17) to obtain that

$$A = \frac{\Gamma(1-\alpha)}{\eta\Gamma(\beta)\alpha}\int_0^\infty y^{\beta-1}e^{-y}\left((\theta+y)^\alpha-(\theta-iu+y)^\alpha\right)dy. \tag{103}$$

The result follows on noting that

$$\int_0^\infty y^{G-1}e^{-y}(M+y)^Y dy = M^{Y+G}\Gamma(G)U(G,Y+G+1,M)$$

from the integral representation of the U function (Abramowitz and Stegun (1972) page 505, 13.2.5). ■

Proof of condition (99):

The condition for obtaining a process of bounded variation on time changing Brownian motion may be expressed in terms of ν by

$$\int_0^\infty dx \int \frac{\nu(dy)}{\sqrt{2\pi y}}(x \wedge 1)\exp(-\frac{x^2}{2y}) < \infty. \tag{A18}$$

Partitioning the integration in (A18) over x into the intervals below and above unity we require that

$$\int \frac{\nu(dy)}{\sqrt{2\pi y}} \left\{ \int_0^1 dx \; x \exp(-\frac{x^2}{2y}) + \int_1^\infty dx \;\; \exp(-\frac{x^2}{2y}) \right\} < \infty.$$

Changing the variable of integration from x to $t = x/\sqrt{y}$ we write that

$$\int \frac{\nu(dy)}{\sqrt{2\pi y}} \left\{ \int_0^{1/\sqrt{y}} ydt \;\, t\exp(-t^2/2) + \int_{1/\sqrt{y}}^\infty \sqrt{y}dt \exp(-t^2/2) \right\} < \infty.$$

Performing the first inner integration we may write

$$\int \frac{\nu(dy)}{\sqrt{2\pi}}\sqrt{y}\left(1 - \exp\left(-\frac{1}{2y}\right)\right) + \int \frac{\nu(dy)}{\sqrt{2\pi}} \int_{1/\sqrt{y}}^\infty dt \exp(-t^2/2) < \infty. \tag{A19}$$

Consider first the second integral in (A19). This may be rewritten as

$$\begin{aligned} \int_0^\infty dt \exp(-t^2/2) \int \frac{\nu(dy)}{\sqrt{2\pi}} \mathbf{1}_{\left(\frac{1}{\sqrt{y}} < t\right)} &= \int_0^\infty dt \exp(-t^2/2) \int \frac{\nu(dy)}{\sqrt{2\pi}} \mathbf{1}_{\left(\frac{1}{t^2} < y\right)} \\ &= \int_0^\infty dt \exp(-t^2/2) \frac{\overline{\nu}(1/t^2)}{\sqrt{2\pi}}, \end{aligned}$$

where $\overline{\nu}(x) = \int_x^\infty \nu(dy)$. This integral is always finite. For the first integral we observe that as

$$\sqrt{y}\left(1 - \exp\left(-\frac{1}{2y}\right)\right) = O(1),$$

as y tends to infinity, this integral is finite near ∞. The condition follows from the behavior of the first integral for y near zero. ■

Proof of Equation (101) and (102):

We use the representation of the Ψ function given by (see Abramowitz and Stegun, page 259, 6.3.22)

$$\Psi(x)+\gamma=\int_0^\infty \frac{e^{-z}-e^{-zx}}{1-e^{-z}}dz$$

to evaluate the Laplace transform (100) at $\lambda=-iu$ and $\lambda=iu$ and multiply the results to obtain that

$$\phi_{\ln(p(t)/p(0))}(u)=\exp\left(\begin{array}{c} -\frac{tu}{\theta}\int_0^\infty \frac{e^{-z}}{1-e^{-z}}\sin\left(\frac{\sqrt{u}}{2}z\right)X \\ \left(\exp\left(\frac{\sqrt{u}}{2}z\right)-\exp\left(-\frac{\sqrt{u}}{2}z\right)\right)dz \end{array}\right)$$

The result follows on writing $(1-e^{-z})^{-1}$ as a power series and integrating each term separately and simplifying.

For the Lévy measure we note that

$$\frac{u^2a}{u^2+a}=\int_0^\infty dxa^2e^{-ax}(1-e^{-u^2x})$$

so taking $a=4n$, and summing over n we obtain:

$$\sum_{n=1}^\infty (4n)^2\exp(-4nx)=\Phi''(x)$$

where

$$\Phi(x)=\sum_{n=1}^\infty \exp(-4nx)=\frac{\exp(-4x)}{1-\exp(-4x)}.$$

■