A gentle introduction to the theory of large cardinals

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In this series of five lectures I will give an introduction to the theory of large cardinals, intended for people with little background in set theory.

Lecture 1. Preliminaries: I will begin with the Zermelo–Fraenkel axioms with Choice (ZFC), and I will recall some fundamental notions from first-order logic and metamathematics, such as completeness, incompleteness, consistency, undecidability, etc. I will also review some basic facts about ordinals and cardinals.

Lecture 2. Small large cardinals: I will introduce the notion of inaccessible cardinal (the smallest of large cardinals), giving some equivalent formulations. I will then move on to Mahlo and weakly-compact cardinals, again giving different equivalent characterizations and showing some applications.

Lecture 3. Measurable cardinals and elementary embeddings: I will introduce the notion of measurable cardinal in terms of measures and prove Scott's Theorem. For this I will need to discuss the notion of elementary embedding and review the construction of models using ultrapowers.

Lecture 4. Larger cardinals: I will introduce strongly-compact and supercompact cardinals and will discuss some of their applications. I will also briefly describe the far reaches of the large-cardinal hierarchy.

Lecture 5. Miscellanea: Depending on the development of the course, and time permitting, I will discuss a variety of issues involving large cardinals, both purely set theoretic and in connection with category theory, abelian groups, etc.

The goal is to provide the necessary background for people who are interested in knowing more about large cardinals and their applications to other areas, such as category theory, algebra, and homotopy theory.

Recommended reading

T. Jech. *Set Theory. The Third Millenium Edition, Revised and Expanded.* Springer, 2003. Chapters 1, 2, 3, 5, 6, 7, 8, 9, 10 of Part I.

A. Kanamori. *The Higher Infinite.* 2nd Edition. Springer, 2003. Chapter 1.

The following general overview of set theory may also be useful in order to put things in perspective:

J. Bagaria. Set Theory. In *The Princeton Companion to Mathematics.* Princeton University Press, 2009.