

SET-THEORETIC ASPECTS OF ABELIAN GROUPS

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I plan to talk only about abelian groups (although much of what I say has analogs for modules over suitable rings) and only in the context of the usual set-theoretic framework, Zermelo–Fraenkel set theory with the axiom of choice (although there are interesting results about what happens in the absence of the axiom of choice).

After some general background information, I plan to concentrate on abelian groups that make a reasonable effort to be free, and I will explain why such efforts sometimes fail (and how the extent of the failure can be measured) and why they sometimes unexpectedly succeed. Both sorts of results depend heavily on set-theoretic considerations. In less vague terms, I intend (if time permits) to describe the following constructions and results:

- **Pontrjagin’s criterion** —essentially saying that countable abelian groups are free if they make a reasonable effort.
- **Powers of the infinite cyclic group** —results of Baer and Specker describe the failure of these groups to be free, and results of Łoś and Eda connect them to large cardinals.
- **Singular compactness** —I may not have time to prove Shelah’s theorem that an abelian group of singular cardinality is free provided all subgroups of smaller cardinality are free, but I hope to at least prove the special case, due to Hill, where the cardinality of the group has countable cofinality. I will also show why similar results hold (for easier reasons) in the case of suitable large cardinals instead of singular ones.
- **Gamma invariants** —a set-theoretic encapsulation of the obstruction to proving freeness of some abelian groups that are exceptionally close to free.

In the (admittedly unlikely) event that some time remains, I will say something about Shelah’s theorem that the Whitehead problem is independent of ZFC.