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Reflection of some properties of uncountable structures

Abstract: Let us consider the following type of reflection principle for a class $\mathcal{C}$ of structures:

($*$) Any $A \in \mathcal{C}$ has the property P if many substructures of A in $\mathcal{C}$ of size $\leq \aleph_1$ have the property P.

Here, "many" in ($*$) above can mean: all, club many, stationarily many, etc. in the set $[A]^{\leq\aleph_1} = \{B \subseteq A : |B| \leq \aleph_1\}$.

It is proved that some of the statements of the form ($*$) are equivalent to the so-called Fodor-type Reflection Principle (FRP) introduced in [2] and hence equivalent to each other over ZFC.

In this talk we shall consider mainly the following two reflection statements (also proved to be equivalent to FRP over ZFC):

(†) A Boolean algebra B is openly generated if club many subalgebras of B of size $\leq \aleph_1$ are openly generated ([3]).

A Boolean algebra B is said to be *openly generated* if, for any σ-closed p.o. P with $\Vdash_P$ "$|B| \leq \aleph_1$", we have $\Vdash_P$ "B is projective". We say that B is *projective* if $B \oplus F$ is free for a sufficiently large free Boolean algebra F. (Note that, by this characterization of openly generatedness, it follows immediately that any Boolean algebra B of cardinality $\leq \aleph_1$ is openly generated if and only if it is projective.)

(‡) A graph $G = \langle G, E\rangle$ has countable coloring number if all subgraphs of G of size $\leq \aleph_1$ have countable coloring number ([4]).

A graph $G = \langle G, E\rangle$ ($E \subseteq G^2$ symmetric and irreflective) has countable coloring number if there is a well-ordering $<$ of G such that, for any $a \in G$, $\{b \in G : b\,E\,a,\ b < a\}$ is finite.

Of course, not everything of the form ($*$) is equivalent to FRP: some of the assertions of the type ($*$) are even false (in ZFC). For example, the assertion obtained by replacing "openly generated" with "free" in (†) is false (see [5]), and the assertion obtained by replacing "coloring number" with "chromatic number" in (†) is also false. We can even prove in ZFC that, for any $\kappa < \beth_\omega$, there is a graph G of uncountable coloring number such that all subgraphs of G of cardinality $\leq \kappa$ have countable coloring number (a theorem by Erdős and Hajnal [1]; see also [6]).

References

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