How to construct large wild algebraic structures

Rüdiger Göbel

This is a summary of my main lectures and the contribution to the workshop.

- **Shelah's elevator —a proof with applications.**
- **The first ω-Erdős cardinal and tree construction inspired by Shelah's 1982 paper.**
- **Absolute E-rings and other absolute structures.**

We know only two distinct methods for constructing large complicated groups, rings and modules, respectively.

1 We may apply directly (diamond-like) prediction principles

to fix the desired properties while the object is still under construction. The key methods are, besides Jensen's diamond or the weak diamond, a long list of prediction principles (due to Shelah), which hold in ordinary set theory of ZFC like the Black Box, the Strong Black Box, the Easy Black Box or (as Menachem would say) the 'silly Black Box', or the most recent $\aleph_n$-Black Box.

2 Implantations

In many cases the desired algebraic properties are already known for simpler objects or are easier to prove for those. Examples are (slightly adjusted) vector spaces or geometries; for instance, trees or lattices. Thus we must develop good encoding techniques such that our desired structures are controlled —like in those horror movies— by their implantations.

In the workshop I will deal with **(1)** and in the three main lectures with **(2)**.

At the first stage I would like to illustrate and prove —and this is real work— the existence of *vector spaces V with five distinguished subspaces* of arbitrary dimension, which are indecomposable in a natural sense. Well, our 'ground fields' can be arbitrary commutative rings R with 1, and V will be a free R-module. Four distinguished subspaces are easy to construct, while the fifth needs work extracted from the Shelah paper in which he solved the Whitehead problem. The second stage is easy: we can construct wild R-modules, e.g., abelian groups.

The second lecture is about more recent work but from the same topic. We now insist that these wild properties hold absolutely; see the appendix for absolute properties. Thus we must disregard the Shelah elevator from the first lecture, because it was based on stationary sets, which are no longer permitted, because they can be destroyed by forcing, i.e., they are not absolute.

Now we encode Shelah's absolute trees [10] into vector spaces V with $\aleph_0$ distinguished subspaces. This is only possible if the dimension of V is less than the first ω-Erdős cardinal $\kappa(\omega)$. Refinement of these methods allows us to show that all cotorsion-free R-algebras A with four primes can be expressed as endomorphism algebras of R-modules absolutely, provided that the cardinals involved are less than $\kappa(\omega)$.

In the third lecture I will discuss the existence of absolute E-rings, which are particularly important for the theory of localizations.

Appendix: The notion of absolute formulas in set theory can be rephrased and slightly extended by saying that a property of an algebraic object holds absolutely if it remains the same in every generic extension of the (set theoretic) universe. Hence, these objects are not effected by forcing arguments. And if we insist on absolute constructions, then many familiar methods of constructing these objects (in particular those involving stationary sets, like the Shelah elevator and Black Box predictions) must be discarded. For further study of these problems, consult the following references on absolute structures.

References

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