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Almost projective modules and Mittag–Leffler conditions

Abstract: Let R be a ring, not necessarily commutative. Following Raynaud and Gruson [4], we say that a right module M over a ring R is Mittag–Leffler provided that the natural map

$$M \otimes \prod_{i \in I} Q_i \longrightarrow \prod_{i \in I} (M \otimes Q_i)$$

is injective for any family of left R-modules $\{Q_i\}_{i \in I}$. A lot of attention has been paid to Mittag–Leffler modules recently; one of the motivations is Drinfeld's paper [3], where Mittag–Leffler modules are proposed as the suitable context for infinite dimensional vector bundles.

In [5] we have proved that the class of flat Mittag–Leffler modules coincides with the class of $\aleph_0$-projective modules. We recall that a module M is said to be $\aleph_0$-projective if it has a family of countably generated projective submodules, closed under unions of countable chains, and such that any countable subset of M is contained in a member of the family. Azumaya and Facchini [1] observed this conection in the case of abelian groups, but it was quite unexpected to find such a functorial characterization of $\aleph_0$-projective modules outside the world of countable rings.

The $\aleph_0$-projective modules played a crucial rôle in Shelah's *solution* of the Whitehead problem, and since then there has been a big effort in constructing large $\aleph_0$-projective modules. Now these ideas are quite useful in the Mittag–Leffler context. For example, using this philosophy we proved in [5] that the class of flat Mittag–Leffler modules is the left class of a cotorsion pair if and only if R is a perfect ring.

Also the theory of Mittag–Leffler modules raises new questions: one of the challenging puzzles is to prove (or disprove!) that the class of flat Mittag–Leffler modules is precovering if and only if the ring R is perfect. This has been shown by Bazzoni and Šťovíček [2] for countable rings. As it has been noticed by Modoi and Šťovíček [6], this type of result produces examples of homotopy categories of complexes that fail to satisfy Brown representability.

References

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