A gentle introduction to the theory of large cardinals

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The language of set theory

The formal language of set theory is the first-order language, with equality, whose only non-logical symbol is the binary relation symbol $\in$. The formulas of the language are defined recursively, as follows:

1. *Atomic formulas* are of the form $x = y$ or $x \in y$.
2. If φ and ψ are formulas, then so are $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$.
3. If φ is a formula, then so are $\forall x \varphi$ and $\exists x \varphi$.

Parentheses may be added after a quantifier to facilitate the reading, and may be omitted if the formula can be read without ambiguity.

The language of set theory

A variable is said to occur free in a formula if it does not fall within the range of any quantifier. Thus x occurs free in the formula $x \in y$, and so does y. The first occurrence of x in the formula $\forall y(x \in y) \wedge \exists x(\neg x \in z)$ is free, while the second is not, as it is bound by the existential quantifier.

A formula with no variables occurring free in it is called a sentence.

The ZFC axioms

We will work in the ZFC (Zermelo-Fraenkel with Choice) axiom system, which is the standard theory of sets. We state the ZFC axioms both informally and formalized in the language of set theory. As is customary, we write $\forall x \in a(\ldots)$ for $\forall x(x \in a \to \ldots)$, and $\exists x \in a(\ldots)$ for $\exists x(x \in a \wedge \ldots)$. The actual formal axioms are the universal closure of the displayed formulas.

Extensionality: If two sets a and b have the same elements, then they are equal.

$$\forall x(x \in a \leftrightarrow x \in b) \to a = b$$

Pair: Given any sets a and b, there exists a set containing a and b as elements.

$$\exists x(a \in x \wedge b \in x)$$

The ZFC axioms

Union: For every set *a*, there is a set containing all elements of the elements of *a*.

$$\exists x \forall y \in a \forall z \in y (z \in x)$$

Power set: For every set *a* there is a set that contains all subsets of *a*.

$$\exists x \forall y (\forall z \in y (z \in a) \to y \in x)$$

Infinity: There exists an infinite set.

$$\exists x (\exists y (y \in x) \land \forall y \in x \exists z \in x (y \in z))$$

Foundation: Every non-empty set *a* contains an $\in$-minimal element.

$$\exists y (y \in a) \to \exists y \in a \forall z \in a (z \notin y)$$

The ZFC axioms

Separation: For every set *a* and every *property*, there is a set containing exactly the elements of *a* that have this *property*.

$$\exists x \forall y (y \in x \leftrightarrow y \in a \wedge \varphi(y))$$

for every formula $\varphi(y)$ of the language of set theory in which x does not occur free and which may have other free variables. So this is an infinite list of axioms, one for each such formula $\varphi(y)$.

Replacement: For every *definable (multivalued) function* on a set *a*, there is a set containing all the values of the *function*.

$$\forall x \in a \exists y \varphi(x, y) \rightarrow \exists z \forall x \in a \exists y \in z \, \varphi(x, y)$$

for every formula $\varphi(x, y)$ of the language of set theory in which z does not occur free and which may have other free variables. This is also an infinite list of axioms, one for each such formula $\varphi(x, y)$.

The ZFC axioms

Choice: For every set *a* of pairwise disjoint non-empty sets, there exists a set that contains exactly one element from each set in *a*.

AC is equivalent, modulo the Zermelo-Fraenkel axioms, to Zermelo's *Well-Ordering Principle*: Every set can be well-ordered. That is, for every set *a* there exists an ordering relation on *a* that is a well-order. (Recall that a well-order of *a* is a linear ordering of *a* in which every non-empty subset of *a* has a least element.)

Sets versus proper classes

Some collections are not sets. For example, the collection of all sets, V, is not a set. Otherwise, by the Separation axiom, there exists a set $A =: \{x \in V : x \notin x\}$. But then $A \in A$ if and only if $A \notin A$. This is known as Russell's Paradox.

Collections that are not sets are called proper classes. In ZFC, proper classes are given by a formula, as in the previous example A was given by the formula $x \notin x$. Another example is the proper class of all sets, V, which is given by the formula $x = x$.

Ordinals

A set *A* is transitive if it contains all elements of its elements.

An ordinal number, or simply an ordinal, is a transitive set well-ordered by $\in$.

The empty set $\emptyset$ is an ordinal.

If α and β are ordinal numbers, then $\alpha \in \beta$ if and only if $\alpha \subset \beta$. Thus, $\alpha \in \beta$ if and only if α is a proper $\in$-initial segment of β. It follows that every ordinal α is precisely the set of all its $\in$-predecessors, which are themselves ordinals.

We usually write $\alpha < \beta$ for $\alpha \subset \beta$, and $\alpha \leq \beta$ for $\alpha \subseteq \beta$. Thus, for all ordinal numbers α and β, either $\alpha < \beta$, or $\beta < \alpha$, or $\alpha = \beta$.

Ordinals

The (immediate) successor of an ordinal α is the ordinal $\alpha \cup \{\alpha\}$, usually denoted by $\alpha + 1$.

If X is a set of ordinals, then $\bigcup X$ is also an ordinal.

The ordinals form a proper class, denoted by Ω or OR, which is well-ordered by $\leq$.

A limit ordinal is an ordinal that is neither empty nor a successor.

The natural numbers are identified with the finite ordinals. Thus, $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, and so on. The set $\mathbb{N}$ of natural numbers is thus identified with the first infinite ordinal number, which is also the first limit ordinal, and is denoted by ω.

Ordinals

An ordinal is countable if it is either finite or bijectable with ω. The set of all countable ordinals is not countable and is, therefore, the first uncountable ordinal, denoted by ω_1. The set of all ordinals bijectable with some $\alpha \leq \omega_1$ is an ordinal not bijectable with any $\alpha \leq \omega_1$ and is denoted by ω_2. And so on.

A limit ordinal α is called regular if there is no function $f : \beta \to \alpha$ with $\beta < \alpha$ and $range(f)$ unbounded in α. Otherwise, α is called singular.

The cofinality of α (denoted by $cof(\alpha)$) is the least $\beta \leq \alpha$ for which there exists $f : \beta \to \alpha$ with range cofinal, i.e., unbounded, in α. Thus, α is regular if and only if $cof(\alpha) = \alpha$.

Ordinals

All the ordinals $\omega, \omega_1, \omega_2, \ldots$ are regular. The limit of all these, that is, $\bigcup_n \omega_n$, is a singular ordinal, denoted by ω_ω.

By the Well-Ordering Principle, every set can be well-ordered. And every well-ordered set X is order-isomorphic to a unique ordinal, denoted by $otp(X)$, the order-type of X.

The universe of all sets

In ZFC, one can prove that the universe of all sets V forms a cumulative hierarchy. That is, every set belongs to some V_α, for some ordinal α, where the V_α are defined as follows:

$V_0 = \emptyset$

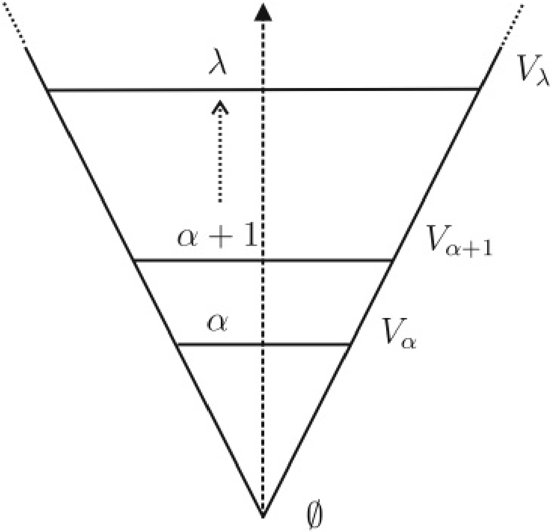
$V_{\alpha+1} = \mathcal{P}(V_\alpha)$, the power set of V_α.

$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$, if λ is a limit ordinal.

Then, $V = \bigcup_{\alpha\in\Omega} V_\alpha$ is the universe of all sets.

Notice that $\alpha \leq \beta$ implies $V_\alpha \subseteq V_\beta$.
One can easily see, by transfinite induction on the ordinals α, that all the V_α are transitive sets.

The universe of all sets



Cardinals

A cardinal number (or simply, a cardinal) is an ordinal that is not bijectable with any smaller ordinal. Thus, all natural numbers are cardinals, and so are ω, ω_1, $\omega_2, \ldots, \omega_\omega, \ldots$
Every infinite cardinal is a limit ordinal.

Given an infinite cardinal κ, the set of all ordinals that are bijectable with some $\lambda \leq \kappa$ is a cardinal; it is the least cardinal greater than κ, and is usually denoted by κ^+. Moreover, if X is a set of cardinals, then $\bigcup X$ is also a cardinal. Hence, the cardinals form a proper class contained in Ω.

Cardinals

The transfinite sequence of all infinite cardinals is denoted, following Cantor, by the Hebrew letter $\aleph$ (aleph) sub-indexed by ordinals. Thus,

$$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\omega, \aleph_{\omega+1}, \ldots, \aleph_\alpha, \ldots$$

Notice that $\aleph_n = \omega_n$, for all $n < \omega$.

The Well-Ordering Principle implies that every set has a cardinality, i.e., is bijectable with a (unique) cardinal $\aleph_\alpha$. The cardinal $\aleph_\alpha$ is called the cardinality of X and is denoted by $|X|$.

Some cardinal arithmetic

Let κ, λ be cardinals.
The sum $\kappa + \lambda$ is defined as $|\kappa \times \{0\} \cup \lambda \times \{1\}|$.
The product $\kappa \cdot \lambda$ is defined as $|\kappa \times \lambda|$.
The exponentiation is defined as $\kappa^\lambda = |\prod_{\alpha<\lambda} \kappa|$. Equivalently, the cardinality of the set of all functions from λ into κ.
Since for every infinite cardinal κ there is a canonical bijection between $\kappa \times \kappa$ and κ, it follows that $\kappa \cdot \kappa = \kappa$, and therefore for all infinite cardinals κ and λ,

$$\kappa + \lambda = \kappa \cdot \lambda = \mathit{max}\{\kappa, \lambda\}.$$

Some cardinal arithmetic

So the sum and product of infinite cardinals is trivial. However, the exponentiation is, in contrast, highly non-trivial. Indeed, even the value of $2^{\aleph_0}$ cannot be decided in ZFC.

If $2 \leq \kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$, because
$2^\lambda \leq \kappa^\lambda \leq (2^\kappa)^\lambda = 2^{\kappa \cdot \lambda} = 2^\lambda$.

Cantor's Theorem states that $|A| > |\mathcal{P}(A)|$, for every set A.
Hence, $2^\kappa > \kappa$, for every cardinal κ.
Another result one can prove in ZFC about infinite cardinal exponentiation is that $\kappa^{cof(\kappa)} > \kappa$, for every infinite cardinal κ.
But, unfortunately, this is almost all one can prove in ZFC about cardinal exponentiation without extra assumptions and assuming, of course, that ZFC is consistent.

Models, consistency, and independence

Since ZFC is a recursive axiom system in which arithmetic is formalizable, it is subject to Gödel's Second Incompleteness Theorem. Namely, if ZFC is consistent, i.e., no contradiction can be logically derived from it, then ZFC cannot prove its own consistency.

However, we do believe ZFC is consistent, since all ZFC axioms are true in V.

Models, consistency, and independence

A structure for the language of set theory is a pair $\langle M, E\rangle$, where M is a set or a proper class and E is a binary relation on M. We say that $\langle M, E\rangle$ is a model of ZFC if all ZFC axioms are *true in* $\langle M, E\rangle$ whenever we interpret the variables as ranging over elements of M and we interpret $\in$ as E. We sometimes consider also models of fragments of ZFC.

A model $\langle M, E\rangle$ is called standard if E is $\in$, that is, the membership relation between sets. Namely, $E = \in \cap (M \times M)$. If $\langle M, E\rangle$ is standard, then we usually write $\in$ instead of E, or we just write M instead of $\langle M, E\rangle$. Thus, V is a standard proper class model of ZFC.

Models, consistency, and independence

By Gödel's Completeness Theorem for first-order logic, ZFC has a model if and only if it is consistent. Hence, one cannot prove in ZFC that there exists a model of ZFC. However, one can prove the existence of models of arbitrarily large finite fragments of ZFC.

The main motivation for building models of (arbitrarily large finite fragments of) ZFC of various sorts is to prove consistency and independence results in mathematics. For suppose φ is a mathematical statement and suppose we can build a model of ZFC where φ holds. Then the negation of φ is not provable in ZFC.
Similarly, if we can build a model of ZFC in which the negation of φ holds, then φ is not provable in ZFC.

Models, consistency, and independence

A sentence φ is said to be independent of ZFC if neither φ not its negation are provable in ZFC. Equivalently, if there exist two models of *ZFC*, one satisfying φ and the other its negation.

The most famous example of independence of ZFC is Cantor's Continuum Hypothesis (CH) (Cantor, 1874): every infinite set of real numbers is either countable (i.e., it can be put into a one-to-one correspondence with the natural numbers) or it has the same cardinality as $\mathbb{R}$ (i.e., it can be put into one-to-one correspondence with the real numbers).
CH is equivalent to $|\mathbb{R}| = \aleph_1$, and also equivalent to $2^{\aleph_0} = \aleph_1$.

Models, consistency, and independence

The CH was Hilbert's first problem in his famous list of 23 unsolved problems he presented at the second International Congress of Mathematicians, held in Paris in 1900.

In spite of many attemps by Cantor himself and others to prove CH, it was not until 60 years later, in 1938, that Gödel constructed his model L, the constructible universe, and proved that CH holds in it, thereby showing that one cannot refute CH in ZFC.

Further, in 1963, Paul Cohen invented a new revolutionary and extremely powerful method for expanding models of ZFC, called forcing, and used it to obtain models of ZFC in which CH fails, thereby showing that one cannot prove CH in ZFC.

The Generalized Continuum Hypothesis (GCH) states that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, for all $\alpha \in \Omega$. The GCH is also independent of ZFC.

The Mostowski collapse

A binary relation E on a set or a proper class X is well-founded if

1. There is no infinite descending E-chain
$$\dots a_{n+1} E a_n \dots a_2 E a_1 E a_0 .$$
   Equivalently, every non-empty subset of X has an E-minimal element.
2. For every $x \in X$, the collection of all $y \in X$ such that yEx is a set. (This, of course, holds automatically if X is a set.)

If E is a well-founded relation on a set (or a proper class) X, then the rank function

$$\rho(x) := sup\{\rho(y) + 1 : yEx\} \tag{1}$$

maps X onto an ordinal, or onto Ω if X is a proper class, and is order-preserving. The function ρ is the unique function satisfying the equation above. $\rho(x)$ is called the rank of x.

The Mostowski collapse

Theorem (Transfinite recursion on well-founded relations)

Suppose E is a well-founded relation on a class X. If G is a class function defined on V, then there is a unique class function F on X such that

$$F(x) = G(x, F \restriction \{z : zEx\}).$$

The Mostowski collapse

Proof.

Call a subset x of X E-transitive if for every $y \in x$, if zEy, then $z \in x$.

Define F as follows:

$F(x) = y$ if and only if there is a function f with domain an E-transitive set containing x such that for every z in the domain of f, $f(z) = G(z, f \restriction \{t : tEz\})$ and $f(x) = y$.

By induction on $\alpha \geq 1$ one can check that F is defined for all $x \in X_\alpha$.

Uniqueness follows by considering another such F', looking at the least α such that the set $\{x \in X_\alpha : F(x) \neq F'(x)\}$ is non-empty, and then taking an E-minimal element x in this set. It follows that $F(x) = F'(x)$, yielding a contradiction. ☐

The Mostowski collapse

A model $\langle M, E\rangle$ is called well-founded if E is well-founded on M.

Theorem (Mostowski collapse)

If $\langle M, E\rangle$ is a well-founded model of the axiom of Extensionality, then there is a unique transitive model $\langle N, \in\rangle$ (called the transitive, or Mostowski, collapse of $\langle M, E\rangle$) and a unique isomorphism $\pi : \langle M, E\rangle \to \langle N, \in\rangle$.

The Mostowski collapse

Proof.

Let $\pi(x) = \{\pi(z) : zEx\}$. Clearly, aEb implies $\pi(a) \in \pi(b)$. So we only need to check that $\pi(a)$ exists for every $a \in M$, and that π is one-to-one. Then we can take N to be the range of π.

Existence is guaranteed by the Transfinite Recursion Theorem applied to E. Indeed, consider the function G such that for each function f with domain an E-transitive set containing x, assigns to the pair $(x, f \restriction \{z : zEx\})$ the *set* $\{f(z) : zEx\}$. Then $\pi(x) = G(x, \pi \restriction \{z : zEx\})$. ☐

The Mostowski collapse

Proof (Continued).

We can see that π is one-to-one by induction on the E-rank ρ of the elements of M. Since M satisfies Extensionality, there is only one element a of M of E-rank 1, hence $\pi(a) = \emptyset$. Now suppose $a, b \in M$ and $a \neq b$. Since M satisfies Extensionality, we can find, say, some cEa such that $\neg cEb$. Hence, $\pi(c) \in \pi(a)$. We claim that $\pi(c) \notin \pi(b)$, and therefore $\pi(a) \neq \pi(b)$. Otherwise, there is dEb with $\pi(c) = \pi(d)$. But since c, d are of lower rank than b, and they are different, by inductive hypothesis we have $\pi(c) \neq \pi(d)$. ☐

Filters

Definition

A filter on a non-empty set A is a set $\mathcal{F}$ of subsets of A such that:

1. $A \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.
2. If $X, Y \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
3. If $X \in \mathcal{F}$ and $X \subseteq Y \subseteq A$, then $Y \in \mathcal{F}$.

If $\mathcal{F}$ is a filter on A, then $\{A - X : X \in \mathcal{F}\}$ is an ideal on A, called the dual ideal of $\mathcal{F}$.

The filter of closed unbounded sets

A subset C of an infinite ordinal α is unbounded if for every $\beta < \alpha$ there is $\gamma \in C$ greater than β. And C is closed if the supremum of every increasing sequence of elements of C belongs to C, provided this supremum is $< \alpha$. Thus, C is closed if and only if for every limit ordinal $\beta < \alpha$, if $C \cap \beta$ is unbounded in β, then $\beta \in C$. We say that C is a cub subset of α if it is closed and unbounded.

If κ is an uncountable cardinal, then the set of limit ordinals smaller than κ is cub. And if λ is a limit cardinal, then the set of cardinals smaller than λ is cub.

The filter of closed unbounded sets

Proposition

If α is an infinite ordinal of uncountable cofinality, then the set $Cub(\alpha) := \{X \subseteq \alpha : C \subseteq X, \text{ for some cub } C\}$ is a filter, called the cub filter on α.

Proof.

This follows easily from the fact that the intersection of any two cub sets is cub. For suppose C and D are cub. Given $\beta < \alpha$, pick alternatively $\gamma_{2n} \in C$ and $\gamma_{2n+1} \in D$ so that

$$\beta < \gamma_0 < \gamma_1 < \ldots < \gamma_{2n} < \gamma_{2n+1} < \ldots$$

Then, $sup\{\gamma_{2n} : n < \omega\} = sup\{\gamma_{2n+1} : n < \omega\} \in C \cap D$, because C and D are closed and α has uncountable cofinality. This shows $C \cap D$ is unbounded. That $C \cap D$ is also closed follows immediately from the fact that both C and D are closed. ☐

The filter of closed unbounded sets

Theorem

If κ is a regular uncountable cardinal, then Cub(κ) is κ-complete, i.e., the intersection of less than κ-many cub sets is cub.

Proof.

Let $\langle C_\alpha : \alpha < \lambda \rangle$, with $\lambda < \kappa$, be a sequence of cub sets subsets of κ. We will prove that $\bigcap_{\alpha<\lambda} C_\alpha$ is cub by induction on λ.
We already saw that the intersection of two cub sets is cub. So we only need to consider the case λ is a limit and assume that the intersection of every sequence of length less than λ of cub sets is cub.
By taking $\bigcap_{\beta\leq\alpha} C_\beta$ instead of C_α, we may assume that the sequence of C_α's is decreasing, i.e., $C_\beta \supseteq C_\alpha$ whenever $\beta \leq \alpha$. $\square$

The filter of closed unbounded sets

Proof (Continued).

Let $C = \bigcap_{\alpha<\lambda} C_\alpha$. Clearly C is closed, since so are all the C_α's. Thus we only need to check that C is unbounded. So fix $\beta < \kappa$. Now define a sequence $\langle \beta_\alpha : \alpha < \lambda \rangle$ as follows: $\beta_0 = \beta$; $\beta_{\alpha+1}$ is the least ordinal in C_α greater than β_α (this is possible because C_α is unbounded); and if α is a limit, then take β_α to be the least ordinal in C_α greater than $sup\{\beta_\gamma : \gamma < \alpha\}$ (this is possible because κ is regular). Then $sup\{\beta_\alpha : \alpha < \lambda\} \in C$. □

The filter of closed unbounded sets

Of course, it is not the case that the intersection of κ-many cub sets is cub. But the diagonal intersection is. Let κ be a regular uncountable cardinal. Given a sequence $\langle X_\alpha : \alpha < \kappa \rangle$ of subsets of κ, the diagonal intersection $\Delta_{\alpha<\kappa} X_\alpha$ is defined as the set $\{\alpha < \kappa : \alpha \in \bigcap_{\beta<\alpha} X_\beta\}$.

Proposition

If κ is a regular uncountable cardinal and $\langle C_\alpha : \alpha < \kappa \rangle$ is a sequence of cub subsets of κ, then $\Delta_{\alpha<\kappa} C_\alpha$ is cub.

The filter of closed unbounded sets

Proof.

Notice first that we may replace C_α by $D_\alpha := \bigcap_{\beta \leq \alpha} C_\beta$, because $\Delta_{\alpha<\kappa} C_\alpha = \Delta_{\alpha<\kappa} D_\alpha$. By Theorem above all the D_α are cub. Note that the sequence of the D_α is decreasing, i.e., $D_\alpha \supseteq D_\beta$ for all $\alpha < \beta < \kappa$.

Now let $C = \Delta_{\alpha<\kappa} D_\alpha$ and let us show that C is cub. Suppose first that $\alpha < \kappa$ is a limit point of C. If $\beta < \alpha$, then every $\gamma \in C$ such that $\beta \leq \gamma < \alpha$ belongs to D_β. Hence since D_β is closed, $\alpha \in D_\beta$. Therefore, $\alpha \in C$.

To see that C is unbounded, fix $\alpha < \kappa$. Construct a sequence $\{\beta_n : n < \omega\}$ as follows. Let $\beta_0 \in D_0$ be greater than α. Given β_n, pick $\beta_{n+1} > \beta_n$ in D_{β_n}. Then let β be the limit of the β_n. We claim that $\beta \in C$. For this it is enough to see that $\beta \in D_\gamma$ for all $\gamma < \beta$. If $\gamma < \beta$, let n be such that $\gamma < \beta_n$. But each β_m, for $m > n$, belongs to D_{β_n}, and so $\beta \in D_{\beta_n} \subseteq D_\gamma$. ☐

Stationary sets

The dual of the cub filter on a cardinal κ of uncountable cofinality is the ideal NS_κ of non-stationary sets.

A subset S of κ is called stationary if it intersects all cub subsets of κ. Thus, every cub set is stationary. Moreover, if S is stationary and C is cub, then $S \cap C$ is stationary.

Stationary sets

A function f on a set of ordinals A is called regressive if $f(\alpha) < \alpha$ for every $\alpha \in A$, $\alpha > 0$.

Theorem (Pressing-Down)

Let κ be a regular uncountable cardinal, and let $S \subseteq \kappa$ be stationary. If $f : S \to \kappa$ is regressive, then there is a stationary $S' \subseteq S$ on which f is constant.

Proof.

Suppose, towards a contradiction, that for every $\alpha < \kappa$, the set $\{\beta \in S : f(\beta) = \alpha\}$ is not stationary. So let $C_\alpha \subseteq \kappa$ be cub and disjoint form the set. Thus, $f(\beta) \neq \alpha$ for every $\beta \in S \cap C_\alpha$. Now let $C = \Delta_{\alpha<\kappa} C_\alpha$. Then $S \cap C$ is stationary and if $\beta \in S \cap C$, then $f(\beta) \neq \alpha$ for all $\alpha < \beta$, contradicting the fact that f is regressive on S. □

Outline

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   - The Levy hierarchy of formulas
   - The Reflection Theorem
   - Inaccessible cardinals
   - Mahlo cardinals
   - Indescribable and weakly-compact cardinals
   - Erdös cardinals

The Levy hierarchy of formulas

A formula in a language that contains the language of set theory is Σ_0, or Π_0, if has only bounded quantifiers $\forall x \in y$ and $\exists x \in y$.

A formula is Σ_1 if it is of the form

$$\exists x_0, \ldots, x_k \varphi(x_0, \ldots, x_k, y_0, \ldots, y_l)$$

where $\varphi(x_0, \ldots, x_k, y_0, \ldots, y_l)$ is Σ_0.

A formula is Π_1 if it is of the form

$$\forall x_0, \ldots, x_k \varphi(x_0, \ldots, x_k, y_0, \ldots, y_l)$$

where $\varphi(x_0, \ldots, x_k, y_0, \ldots, y_l)$ is Σ_0.

The Levy hierarchy of formulas

In general, a formula is Σ_n, $n > 1$ if it is of the form

$$\exists x_0, \ldots, x_k \varphi(x_0, \ldots, x_k, y_0, \ldots, y_l)$$

where $\varphi(x_0, \ldots, x_k, y_0, \ldots, y_l)$ is Π_{n-1}.

And a formula is Π_n, $n > 1$, if it is of the form

$$\forall x_0, \ldots, x_k \varphi(x_0, \ldots, x_k, y_0, \ldots, y_l)$$

where $\varphi(x_0, \ldots, x_k, y_0, \ldots, y_l)$ is Σ_{n-1}.

The Levy hierarchy of formulas

Σ_1 formulas are upwards absolute for transitive sets or classes. That is, if $M \subseteq N$ are transitive, $\varphi(x)$ is a Σ_1 formula, and $a \in M$ is such that $\varphi(a)$ is true in M, written $M \models \varphi(a)$, then $N \models \varphi(a)$. (Exercise.) Similarly, Π_1 formulas are downwards absolute for transitive sets or classes, that is, if $\varphi(x)$ is Π_1, $a \in M$, and $N \models \varphi(a)$, then $M \models \varphi(a)$.

If in a formula $\varphi(x_0, \ldots, x_k)$, where $x_0, \ldots, x_k$ occur free, we fix the values $a_0, \ldots, a_k$ of the variables $x_0 \ldots, x_k$, then we say that $\varphi(a_0, \ldots, a_k)$ is a formula with parameters $a_0, \ldots, a_k$.

The Reflection Theorem

For every natural number n, we have the following.

Theorem (A. Levy, 1960)

There is a cub proper class C_n of cardinals such that for every $\kappa \in C_n$,

$$V_\kappa \prec_n V$$

i.e., for all $\kappa \in C_n$, all $a \in V_\kappa$ and all $\varphi(x) \in \Sigma_n$,

$$V_\kappa \models \varphi(a) \quad \textit{if and only if} \quad V \models \varphi(a)$$

The Reflection Theorem

Proof.

For $n = 0$ this is clear, since we may take C_0 to be the class of all cardinals. So suppose we have proved the Theorem for n, and so we have C_n.
Given $\alpha \in C_n$, let $f(\alpha) \in C_n$ be the least cardinal such that for every formula $\exists x \varphi(x, x_1, ..., x_k)$, where φ is Π_n, and every $a_1, ..., a_k$ in V_α, if $\exists x \varphi(x, a_1, ..., a_k)$, then $\varphi(b, a_1, ..., a_k)$ for some $b \in V_{f(\alpha)}$. For each $n < \omega$, let $f^n(\alpha)$ be the n-iterate of f at α. Let $F(\alpha)$ be the limit of all $f^n(\alpha)$, $n < \omega$. Thus $F(\alpha)$ is a cardinal (and it belongs to C_n). Then $C_{n+1} = \{F(\alpha) : \alpha \in C_n\}$ is as required. ☐

Notice that for every ordinal α we have $V_\alpha \prec_0 V$.

One may naturally wonder whether there can be a *regular* cardinal κ such that $V_\kappa \prec_1 V$. This leads us to the first of the large cardinals.

Inaccessible cardinals

A cardinal κ is (strongly) inaccessible if it is uncountable, regular, and a strong limit, i.e., for every cardinal $\lambda < \kappa$, $2^{\lambda} < \kappa$.

If κ is inaccessible, then $|V_{\kappa}| = \kappa$ and $\kappa = \aleph_{\kappa}$.

We shall see next that κ is inaccessible if and only if it is regular and V_{κ} is a model of ZFC. It follows, by Gödel's Second Incompleteness Theorem, that one cannot prove in ZFC that inaccessible cardinals exist.

Elementary substructures

We will sometimes consider the language of set theory enriched with additional relation, function, or constant symbols, as well as the corresponding structures for these languages. E.g., structures of the form $\langle M, \in, A, a\rangle$, where A is a subset of M and $a \in M$.

Given any two structures $M \subseteq N$ in a given language, we write $M \prec_n N$ if M is a Σ_n-*elementary substructure* of N, i.e., for every Σ_n formula $\varphi(x_0, \ldots, x_k)$ and every $a_0, \ldots, a_k \in M$,

$$M \models \varphi(a_0, \ldots, a_k) \text{ if and only if } N \models \varphi(a_0, \ldots, a_k).$$

M is an elementary substructure of N, written $M \prec N$, if $M \prec_n N$ for all n. Thus, if $M \prec N$, then M and N satisfy the same sentences.

The Löwenheim-Skolem Theorem I

The Löwenheim-Skolem Theorem for first-order logic asserts that for every infinite cardinal κ, every structure M for a countable language, and every $X \subseteq M$ of cardinality κ, there is an elementary substructure N of M with $X \subseteq N$ and such that N has cardinality κ. In particular, every structure M for a countable language has a countable elementary substructure.

Such a structure N can be obtained by closing X under a family of Skolem functions, one for each existential formula. The resulting structure N is called called the Skolem Hull of X.

The Löwenheim-Skolem Theorem

More precisely, for each existential formula $\exists x \varphi(x, y_1, \ldots, y_n)$, one has a function $f : M^n \to M$ that assigns to each n-tuple $\langle a_1, \ldots, a_n \rangle$ a witness to the sentence $\exists x \varphi(x, a_1, \ldots, a_n)$, whenever the sentence holds in M, and some fixed element of M otherwise. Every well-ordering of M gives rise to a family of definable Skolem functions, namely, $f(a_1, \ldots, a_n)$ is defined as *the least* witness to $\exists x \varphi(x, a_1, \ldots, a_n)$ under the well-ordering.

An application of the Löwenheim-Skolem Theorem

Every set is contained in a smallest transitive set, called its transitive closure. The transitive closure of a set A, denoted by $TC(A)$ consists of all elements of A, the elements of elements of A, the elements of elements of elements of A, and so on.

Suppose M is a transitive model of some fragment $T \subseteq ZFC$, and suppose $A \in M$ is of infinite cardinality λ. Then we can take the Skolem Hull of A in M, call it N. Thus $N \models T$, $A \subseteq N$, and N has cardinality λ.
Since N satisfies Extensionality, because M does, being transitive, we can take the Mostowski collapse $\bar{N}$ of N. Thus $\bar{N} \models T$ and $\bar{N}$ has cardinality λ. Moreover, if $TC(\{a\}) \subseteq A$, then the Mostowski collapse of a is a itself, and so $a \in \bar{N}$.

Some characterizations of inaccessible cardinals

Theorem

The following are equivalent for a regular cardinal κ:

1. κ *is inaccessible.*
2. $V_\kappa \models ZFC$.
3. $V_\kappa \prec_1 V$.

Proof.

(1) implies (2): Let us check that V_κ satisfies Replacement. So, suppose F is a class function in V_κ whose domain is an element of V_κ. Thus, F has cardinality less than κ, and since κ is regular, F is not cofinal in V_κ and so it is contained in some V_α, $\alpha < \kappa$. But then the range of F belongs to $V_{\alpha+1}$. ☐

Some characterizations of inaccessible cardinals

Continued.

(2) implies (3): Let $\exists x \psi(x, a)$ be a Σ_1 sentence, with parameter $a \in V_\kappa$, and suppose $\exists x \psi(x, a)$ holds. Notice that since $V_\kappa \models ZFC$, $|TC(\{a\})| < \kappa$. Let b be a witness to $\exists x \psi(x, a)$ and let λ be a regular cardinal greater than κ such that $b \in V_\lambda$. Let N be an elementary substructure of V_λ with $b \in N$, $TC(\{a\}) \subseteq N$ and has cardinality $< \kappa$. Let M be the Mostowski collapse of N. Let c be the collapse of b. Since a collapses to itself, $M \models \psi(c, a)$. ☐

Some characterizations of inaccessible cardinals

Proof (Continued).

Hence, since Σ_1 sentences are upwards-absolute for transitive models, $V_\kappa \models \exists x \psi(x, a)$.

(3) implies (1): We check that κ is strong limit. So, suppose λ is a cardinal less than κ. Then, $\exists \alpha \exists f (f : \alpha \to V_{\lambda+1}$ is onto). But this is a Σ_1 sentence with $V_{\lambda+1}$ as a parameter and so it holds in V_κ. □

More on inaccessible cardinals I

Theorem (Levy, 1960)

A cardinal κ is inaccessible if and only if for every $A \subseteq V_\kappa$ there is a $\lambda < \kappa$ (equivalently, a cub set of λs) such that

$$\langle V_\lambda, \in, A \cap V_\lambda \rangle \prec \langle V_\kappa, \in, A \rangle .$$

More on inaccessible cardinals II

Proof.

Suppose κ is inaccessible and let $A \subseteq V_\kappa$. Build a chain of elementary substructures of $\langle V_\kappa, \in, A\rangle$, each structure in the chain of size $< \kappa$, so that the union of the chain is of the form $\langle V_\lambda, \in, A \cap V_\lambda\rangle$, some $\lambda < \kappa$.

For the other direction, suppose κ is singular. Let A be a function whose domain is some $\mu < \kappa$ and whose range is cofinal on κ. Let $\lambda > \mu$ be such that $\langle V_\lambda, \in, A \cap V_\lambda\rangle \prec \langle V_\kappa, \in, A\rangle$. Then, the range of A is contained in λ, which is impossible.

That κ is a strong limit is shown by a similar argument. □

Mahlo cardinals

If κ is inaccessible, then the set C of all strong limit cardinals smaller than κ is cub. So if κ is the least inaccessible cardinal, then all cardinals in C must be singular, for otherwise there would be an inaccessible cardinal below κ.

An inaccessible cardinal κ is called Mahlo (after Paul Mahlo, German mathematician) if the set of inaccessible cardinals smaller than κ is stationary. Thus κ is Mahlo if and only if it is inaccessible and every cub subset of κ contains an inaccessible cardinal.

One cannot prove from ZFC plus the existence of an inaccessible cardinal that a Mahlo cardinal exists. For suppose $\kappa < \lambda$ are the first two inaccessible cardinals. Then V_λ is a model of ZFC which satisfies “There exists an inaccessible cardinal" plus “There is no Mahlo cardinal".

Mahlo cardinals

Theorem (Levy, 1960)

A cardinal κ is Mahlo if and only if for every $A \subseteq V_\kappa$ there is a regular (equivalently, an inaccessible) cardinal $\lambda < \kappa$ (equivalently, a stationary set of λs) such that

$$\langle V_\lambda, \in, A \cap V_\lambda \rangle \prec \langle V_\kappa, \in, A \rangle.$$

Proof.

Similarly as in the case of inaccessible cardinals. For the if direction, suppose C is a cub subset of κ. Let $\lambda < \kappa$, λ inaccessible, be such that

$$\langle V_\lambda, \in, C \cap V_\lambda \rangle \prec \langle V_\kappa, \in, C \rangle.$$

Then C is unbounded in λ. Hence, $\lambda \in C$. ☐

Indescribable and weakly-compact cardinals

Pushing the reflection principles a bit further, we can ask: Why should we restrict to first-order logic?

In second-order logic we have two kinds of variables: first-order variables $x, y, z, \ldots$, and second-order variables $X, Y, Z \ldots$, which may also be quantified. We also have predicates $X(x)$. Second-order variables are interpreted in a given structure $\langle M, \ldots \rangle$ as subsets of M, and the predicates $X(x)$ are interpreted as $x \in X$.

Indescribable and weakly-compact cardinals

A second order formula is called Σ^1_0 (or Π^1_0) if its quantifiers range only over variables of first order, but it may have free variables of second order.

A formula is Σ^1_1 if it is of the form

$$\exists X_0,\ldots,X_k\varphi(X_0,\ldots,X_k,Y_0,\ldots,Y_l)$$

where $\varphi(X_0,\ldots,X_k,Y_0,\ldots,Y_l)$ is Σ^1_0.

A formula is Π^1_1 if it is of the form

$$\forall X_0,\ldots,X_k\varphi(X_0,\ldots,X_k,Y_0,\ldots,Y_l)$$

where $\varphi(X_0,\ldots,X_k,Y_0,\ldots,Y_l)$ is Σ^1_0.

Indescribable and weakly-compact cardinals

Notice that κ is inaccessible iff for every $A \subseteq V_\kappa$ and every Σ^1_0 sentence φ in the language of set theory with one additional predicate symbol for A, if $\langle V_\kappa, \in, A\rangle \models \varphi$, then for some $\lambda < \kappa$, $\langle V_\lambda, \in, A \cap V_\lambda\rangle \models \varphi$.

We say that κ is Σ^1_1-indescribable (Π^1_1-indescribable) if for every $A \subseteq V_\kappa$ and every Σ^1_1 (Π^1_1) sentence φ in the language of set theory with one additional predicate symbol for A, if $\langle V_\kappa, \in, A\rangle \models \varphi$, then there is $\lambda < \kappa$ such that $\langle V_\lambda, \in, A \cap V_\lambda\rangle \models \varphi$.

Indescribable and weakly-compact cardinals

We actually have the following characterization of inaccessibility.

Proposition

κ is Σ^1_1-indescribable iff it is inaccessible.

However, Π^1_1-indescribability leads to the next large-cardinal notion.

Weakly-compact cardinals

Weakly-compact cardinals were studied by Paul Erdös and Alfred Tarski in the context of the partition calculus. Namely, κ is weakly-compact if κ is an uncountable cardinal and satisfies $\kappa \to (\kappa)^2$, i.e., for every coloring of all pairs of elements of κ with two colors, there is a subset X of κ of cardinality κ such that every pair of elements of X has the same color. Thus, weak-compactness generalizes Ramsey's theorem to the uncountable.

Weakly-compact cardinals

Lemma

If κ is weakly-compact, then κ is inaccessible.

Proof.

Suppose $\kappa = \bigcup\{X_\alpha : \alpha < \lambda\}$, where all the X_α are pairwise disjoint, $\lambda < \kappa$ and $|X_\alpha| < \kappa$, all $\alpha < \lambda$. Let *f* be the coloring given by: $f(\{\beta, \gamma\}) = 1$ iff β and γ belong to the same X_α. Then *f* has no homogeneous set of size κ. This shows κ is regular. ☐

Weakly-compact cardinals

Continued.

To see that κ is a strong limit, suppose, towards a contradiction, that $\{g_\alpha : \alpha < \kappa\}$ is a collection of functions from a fixed $\lambda < \kappa$ into 2. Let f be the coloring given by: $f(\{\alpha, \beta\}) = 1$ iff $g_\alpha <_{lex} g_\beta$ iff $\alpha < \beta$.

An f-homogeneous set produces an increasing or a decreasing sequence under the lexicographic ordering. But it is a general fact that there cannot be any such sequence of length λ^+: for suppose $\{h_\alpha : \alpha < \lambda^+\}$ is an increasing sequence. Let $\gamma \leq \lambda$ be the least ordinal such that $\{h_\alpha \restriction \gamma : \alpha < \lambda^+\}$ has size λ^+. So, we may assume all the $h_\alpha \restriction \gamma$ are distinct. For each α, let δ_α be the least ordinal where h_α and $h_{\alpha+1}$ differ. Note that $\delta_\alpha < \gamma$. So, we may assume all δ_α are the same, call it δ. But if $h_\alpha \restriction \delta = h_\beta \restriction \delta$, then $h_\beta <_{lex} h_{\alpha+1}$ and $h_\alpha <_{lex} h_{\beta+1}$. Hence, $\alpha = \beta$. Thus, $\{h_\alpha \restriction \delta : \alpha < \lambda^+\}$ has size λ^+, contradicting the minimality of γ. ☐

Aronszajn trees

An Aronszajn κ-tree is a tree of height κ with levels of size $< \kappa$ and with no branch of size κ.

The following is a useful characterization of weakly-compact cardinals.

Theorem

κ is weakly-compact iff it is inaccessible and there are no Aronszajn κ-trees.

A characterization of weakly-compact cardinals

Theorem (Hanf and Scott 1961; Keisler 1962)

The following are equivalent for a cardinal κ:

1. κ *is* Π^1_1*-indescribable.*
2. κ *is weakly-compact.*
3. *For every* $A \subseteq V_\kappa$*, there is a transitive set* M *with* $\kappa \in M$ *and* $X \subseteq M$ *such that* $\langle V_\kappa, \in, A\rangle \prec \langle M, \in, X\rangle$*.*

Stationary Reflection

Theorem (Stationary Reflection)

If κ is weakly compact, then for every stationary subset S of κ, there exists an inaccessible λ such that $S \cap \lambda$ is stationary.

Proof.

Let $F : V_\kappa \to \kappa$ be such that if λ is a cardinal, then $F(\lambda) = 2^\lambda$, and if f is a function from some ordinal α into κ, then $F(f) = sup(range(f))$. Such an F exists because κ is inaccessible. ☐

Stationary Reflection

Continued.

The sentence: “S is stationary" can be expressed as a Π^1_1 sentence over $\langle V_\kappa, \in, S, F\rangle$. Indeed,

$$\forall C(C \text{ is cub} \rightarrow \exists \alpha \in C(\alpha \in S)).$$

And the sentence : “For every function $f : \alpha \rightarrow \kappa$, $F(f)$ exists" can also be expressed as a Π^1_1 sentence over $\langle V_\kappa, \in, S, F\rangle$. Namely,

$$\forall f(\exists \alpha(\alpha \in OR \wedge dom(f) = \alpha) \wedge range(f) \subseteq OR \rightarrow \exists \beta F(f) = \beta).$$

☐

Stationary Reflection

Continued.

Since κ is Π^1_1-indescribable, there exists $\lambda < \kappa$ such that $\langle V_\lambda, \in, S \cap V_\lambda, F \cap V_\lambda \rangle$ satisfies

$$\forall C(C \text{ is cub} \to \exists \alpha \in C(\alpha \in S \cap V_\lambda))$$

and also

$$\forall f(\exists \alpha(\alpha \in OR \wedge dom(f) = \alpha) \wedge range(f) \subseteq OR \to \exists \beta F \cap V_\lambda(f) = \beta).$$

The first sentence implies that $S \cap \lambda$ is stationary in λ. And the second sentence that λ is regular. Finally, since V_λ is closed under F, λ must be a strong limit cardinal. □

Stationary Reflection

Corollary

Every weakly-compact cardinal is Mahlo.

Proof.

Let C be a cub subset of κ. By the Theorem, there is λ inaccessible such that $C \cap \lambda$ is stationary in λ. hence $\lambda \in C$. ☐

Erdös cardinals

A strengthening of $\kappa \to (\kappa)^2$, or rather its equivalent form: for every $n < \omega$, $\kappa \to (\kappa)^n$, would be to require the existence of sets that are simultaneously homogeneous for all $n < \omega$. Namely, for X a set, let $[X]^{<\omega}$ be the set of all finite subsets of X. For α an ordinal and κ a cardinal, the notation $\kappa \to (\alpha)^{<\omega}$ means that for every coloring of $[\kappa]^{<\omega}$ into two colors, there is a homogeneous set of order-type α, i.e., a subset X of κ of order-type α such that for every n, all elements of $[X]^n$ have the same color. Notice that we cannot require that all elements of $[X]^{<\omega}$ have the same color, since we could color $[\kappa]^n$ all green and $[\kappa]^m$ all red, for different n and m.

If $\alpha \geq \omega$, the α-Erdös cardinal is the least cardinal κ such that $\kappa \to (\alpha)^{<\omega}$. We denote such a κ, if it exists, by $\kappa(\alpha)$.

Erdös cardinals

Erdös cardinals can be characterized in terms of indiscernibles. Namely,

Lemma (J. H. Silver)

For $\alpha \geq \omega$, we have $\kappa \to (\alpha)^{<\omega}$ iff for every structure M in a countable language with $\kappa \subseteq M$, there is a set $X \subseteq \kappa$ of order-type α of M-indiscernibles. i.e., for every formula $\varphi(x_1, ..., x_n)$ in the language of M, and every $\alpha_1 < ... < \alpha_n$ and $\beta_1 < ... < \beta_n$ in X,

$$M \models \varphi(\alpha_1, ..., \alpha_n) \text{ iff } M \models \varphi(\beta_1, ..., \beta_n).$$

Erdös cardinals

Proof.

Let $\{\varphi_n : n < \omega\}$ be an enumeration of all the formulas of the language of M so that φ_n has at most n free variables. Let $f : [\kappa]^{<\omega} \to 2$ be given by: $f(\alpha_1, ..., \alpha_n) = 0$ iff $M \models \varphi_n(\alpha_1, ..., \alpha_n)$. Then any f-homogeneous set of order-type α is a set of M-indiscernibles.

Conversely, if $f : [\kappa]^{<\omega} \to 2$ and X is a set of indiscernibles for the structure $\langle \kappa, \in, f \restriction [\kappa]^n \rangle_{n \in \omega}$, then X is f-homogeneous. $\square$

Erdös cardinals

How large are Erdös cardinals? It is not very hard to see that $\kappa(\omega)$ is Π^1_1-describable and so it is not weakly-compact. It can be shown, however, that $\kappa(\omega)$ is inaccessible. Even though $\kappa(\omega)$ itself has not very strong large-cardinal properties, there are very large cardinals below it.

Theorem (Reinhardt and Silver)

There is a totally indescribable cardinal below $\kappa(\omega)$*.*

Outline

3. Lecture III
    - Ultrafilters
    - κ-complete ultrafilters
    - Measurable cardinals

Ultrafilters

Definition

A filter $\mathcal{F}$ on a set A is called an ultrafilter if for every $X \subseteq A$, either $X \in \mathcal{F}$ or $A - X \in \mathcal{F}$.

An ultrafilter $\mathcal{F}$ on a set A is called principal if and only if there exists $a \in A$ such that $\mathcal{F} = \{X \subseteq A : a \in X\}$.
Every filter on a finite set A is principal.
An example of a non-principal filter on ω is the *Fréchet filter*, which is the set of all co-finite subsets of ω, i.e., $\{X \subseteq \omega : \omega - X \text{ is finite}\}$. More generally, if κ is an infinite cardinal, then the set of all subsets of κ whose complement has cardinality less than κ is a filter.

κ-complete ultrafilters

Let κ be an infinite cardinal. A filter $\mathcal{F}$ on a set A is called κ-complete if the intersection of less than κ-many elements of $\mathcal{F}$ belongs to $\mathcal{F}$. ω_1-complete filters are also called σ-complete.

The filter $\{X \subseteq \omega_1 : |\omega_1 - X| \leq \aleph_0\}$ is σ-complete. More generally, for every uncountable regular cardinal κ, the filter $\{X \subseteq \kappa : |\kappa - X| < \kappa\}$ is κ-complete. The filter of subsets of $[0, 1]$ of Lebesgue measure 1 is σ-complete.

A natural question is if there exists a σ-complete non-principal ultrafilter on some set A, equivalently on some cardinal κ.

κ-complete ultrafilters

Proposition

Suppose $\lambda \leq \kappa$ are infinite cardinals. An ultrafilter $\mathcal{F}$ on κ is λ-complete if and only if for every partition $\{X_\alpha : \alpha < \mu\}$ of κ, where $\mu < \lambda$, there exists α such that $X_\alpha \in \mathcal{F}$.

Proof.

$\Rightarrow$. Suppose $\{X_\alpha : \alpha < \mu\}$, some $\mu < \lambda$, is a partition of κ. If none of the X_α's is in $\mathcal{F}$, then $\kappa - X_\alpha \in \mathcal{F}$, for all $\alpha < \mu$. Hence by λ-completeness, $\bigcap_{\alpha<\mu}(\kappa - X_\alpha) = \emptyset \in \mathcal{F}$, which is impossible.
$\Leftarrow$. By induction on λ. So assume $\mathcal{F}$ is λ-complete and let us show that it is λ^+-complete. □

κ-complete ultrafilters

Continued.

Given $\{X_\alpha : \alpha < \lambda\} \subseteq \mathcal{F}$, let $Y_0 = X_0$, let $Y_{\alpha+1} = Y_\alpha \cap X_{\alpha+1}$, and for α limit let $Y_\alpha = \bigcap_{\beta<\alpha} Y_\beta$. By the inductive assumption, all Y_α belong to $\mathcal{F}$.
Now let $Z_\alpha = Y_\alpha - Y_{\alpha+1}$. Thus,

$$\{\kappa - X_0\} \cup \{Z_\alpha : \alpha < \lambda\} \cup \{\bigcap_{\alpha<\lambda} Y_\alpha\}$$

is a partition of κ.
Since $X_0 \in \mathcal{F}$, $\kappa - X_0 \notin \mathcal{F}$. And $Z_\alpha \notin \mathcal{F}$ for all α, because $\kappa - Z_\alpha = \kappa - (Y_\alpha - Y_{\alpha+1}) = (\kappa - Y_\alpha) \cup Y_{\alpha+1} \in \mathcal{F}$. Hence by our assumption,

$$\bigcap_{\alpha<\lambda} Y_\alpha = \bigcap_{\alpha<\lambda} X_\alpha \in \mathcal{F}.$$

☐

κ-complete ultrafilters

It follows that if $\mathcal{U}$ is a κ-complete ultrafilter on κ and $\bigcup_{\alpha<\lambda} X_\lambda \in \mathcal{U}$, where $\lambda < \kappa$, then $X_\alpha \in \mathcal{U}$ for some $\alpha < \lambda$.

Proposition

If κ is the least cardinal for which there exists a non-principal σ-complete ultrafilter $\mathcal{F}$ on κ, then $\mathcal{F}$ is in fact κ-complete.

κ-complete ultrafilters

Proof.

Notice that the assumption implies κ is uncountable. So, suppose, to the contrary, that $\{X_\alpha : \alpha < \lambda\}$, some infinite cardinal $\lambda < \kappa$, is a partition of κ such that $X_\alpha \notin \mathcal{F}$, for all $\alpha < \lambda$. Then define the filter $\mathcal{G}$ on λ as follows

$$X \in \mathcal{G} \text{ if and only if } \bigcup_{\alpha \in X} X_\alpha \in \mathcal{F}.$$

$\mathcal{G}$ is non-principal, for if $\alpha < \lambda$ is such that $G = \{X \subseteq \lambda : \alpha \in X\}$, then $\{\alpha\} \in G$, and therefore $X_\alpha \in \mathcal{F}$, which is impossible. ☐

κ-complete ultrafilters

Continued.

We claim that $\mathcal{G}$ is an ultrafilter, for if $X \subseteq \lambda$ is not in $\mathcal{G}$, then $\bigcup_{\alpha \in X} X_\alpha \notin \mathcal{F}$. And since $\mathcal{F}$ is an ultrafilter this implies

$$\kappa - \bigcup_{\alpha \in X} X_\alpha = \bigcap_{\alpha \in X} (\kappa - X_\alpha) = \bigcap_{\alpha \in X} \bigcup_{\beta \neq \alpha} X_\beta = \bigcup_{\alpha \in (\lambda - X)} X_\alpha \in \mathcal{F}$$

hence $\lambda - X \in \mathcal{G}$.

Suppose now that $\{Y_n : n < \omega\} \subseteq \mathcal{G}$. Then, $\bigcup_{\alpha \in Y_n} X_\alpha \in \mathcal{F}$, for every n. Since $\mathcal{F}$ is σ-complete,

$$\bigcap_{n<\omega} \bigcup_{\alpha \in Y_n} X_\alpha = \bigcup_{\alpha \in \bigcap_{n<\omega} Y_n} X_\alpha \in \mathcal{F}$$

and so $\bigcap_{n<\omega} Y_n \in \mathcal{G}$. □

Measurable cardinals

A uncountable cardinal κ is called measurable if there exists a κ-complete non-principal ultrafilter on κ.

By the last Proposition, if κ is the least cardinal on which there exists a σ-complete non-principal ultrafilter, then κ is measurable.

Elementary embeddings

If N and M are structures for the language of set theory, a function $j : N \to M$ is an elementary embedding if for every formula $\varphi(x_1, ..., x_n)$ and every $a_1, ..., a_n \in N$,

$$N \models \varphi(a_1, ..., a_n) \text{ iff } M \models \varphi(j(a_1), ..., j(a_n)).$$

Elementary embeddings

Suppose now that $M \subseteq N$ are models of ZFC, with N transitive, and $j : N \to M$ is an elementary embedding which is not the identity. Then there is a least ordinal α that is moved by j. To see this, let x be a set in N of least rank such that $j(x) \neq x$. Let $\alpha = rank(x)$. Since the elements of x have rank smaller than α, $x \subseteq j(x)$. So there is $y \in j(x) \setminus x$. But then $\alpha \leq rank(y)$, since otherwise $j(y) = y \in j(x)$, and therefore by elementarity of j, $y \in x$, which is not the case. Thus, $\alpha \leq rank(y) < rank(j(x)) = j(\alpha)$.

The least ordinal α moved by j is called the critical point of j, denoted by $crit(j)$.

Elementary embeddings

Proposition

If $\alpha = crit(j)$, then α is a regular cardinal in N.

Proof.

Let us show first that α is a cardinal. Otherwise, there is $\beta < \alpha$ and a bijection $f : \beta \to \alpha$. But then, by elementarity, $j(f) : \beta \to j(\alpha)$ is also a bijection, which is impossible because $f(\gamma) = j(f)(\gamma)$ for all $\gamma < \beta$. A similar argument shows α is regular. □

The ultrapower construction

Given an ultrafilter $\mathcal{U}$ on some cardinal κ we can form the ultrapower of V by $\mathcal{U}$, denoted by $Ult(V,\mathcal{U})$, as follows. Let V^κ be the proper class of all κ-sequences of sets. We define an equivalence relation $\equiv_{\mathcal{U}}$ on V^κ by:

$$f \equiv_{\mathcal{U}} g \text{ if and only if } \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in \mathcal{U}.$$

Since the equivalence classes $[f]$ are proper classes, we redefine

$$[f] := \{g : g \equiv_{\mathcal{U}} f \text{ and } \forall h (h \equiv_{\mathcal{U}} f \to rank(g) \leq rank(h))\}$$

which is a set.

The ultrapower construction

Now define a relation $E_{\mathcal{U}}$ on $V^\kappa / \equiv_{\mathcal{U}}$ by:

$$[f] E_{\mathcal{U}} [g] \text{ if and only if } \{\alpha < \kappa : f(\alpha) \in g(\alpha)\} \in \mathcal{U}.$$

The ultrapower $Ult(V, \mathcal{U})$ is defined as $\langle V^\kappa / \equiv_{\mathcal{U}}, E_{\mathcal{U}} \rangle$.
It is not hard to check (Łoś Theorem) that

$$Ult(V, \mathcal{U}) \models \varphi([f_1], \ldots, [f_n]) \text{ iff } \{\alpha < \kappa : \varphi(f_1(\alpha), \ldots, f_n(\alpha))\} \in \mathcal{U}.$$

The ultrapower construction

If φ is a sentence in the language of set theory, then $Ult(V,\mathcal{U}) \models \varphi$ if and only if $V \models \varphi$. Thus, V and $Ult(V,\mathcal{U})$ are elementarily equivalent.

For each x, let c_x be the function on κ with constant value x. Then, the map $j : V \to Ult(V,\mathcal{U})$ given by $j(x) = [c_x]$ is an elementary embedding.

The ultrapower construction

Proposition

If $\mathcal{U}$ is σ-complete, then $Ult(V,\mathcal{U})$ is well-founded.

Proof.

First notice that for every $[f] \in Ult(V,\mathcal{U})$, the collection of all $[g]$ such that $[g]E_{\mathcal{U}}[f]$ is a set, because for each such g there is $h \in [g]$ with $rank(h) \leq rank(f)$.

Now suppose, towards a contradiction, that there is an infinite descending chain $[f_{n+1}]E_{\mathcal{U}}[f_n]$. For each n, let $X_n \in \mathcal{U}$ witness $[f_{n+1}]E_{\mathcal{U}}[f_n]$. By σ-completeness, there is $\alpha \in \bigcap_{n<\omega} X_n$. But then, $f_{n+1}(\alpha) \in f_n(\alpha)$, for all n, thus giving an infinite descending $\in$-chain, which is impossible. ☐

Measurable cardinals and elementary embeddings

Theorem (Keisler and Scott, 1961)

κ is measurable if and only if there exists an elementary embedding $j : V \to M$, with M transitive, such that $\kappa = crit(j)$.

Proof.

Suppose first that κ is measurable, and let $\mathcal{U}$ be a κ-complete non-principal ultrafilter over κ. Let $j_{\mathcal{U}} : V \to Ult(V, \mathcal{U})$ be the corresponding elementary embedding. The ultrapower $Ult(V, \mathcal{U})$ is well-founded, so there is a Mostowski collapse class isomorphism $\pi : Ult(V, \mathcal{U}) \to M$, with M transitive. Then the embedding $j := \pi \circ j_{\mathcal{U}} : V \to M$ is elementary, so we only need to check that $\kappa = crit(j)$. □

Measurable cardinals and elementary embeddings

Proof (Continued).

Let $\gamma < \kappa$ and assume $j(\beta) = \beta$ for all $\beta < \gamma$. If $\gamma < j(\gamma)$, then $[f]E_{\mathcal{U}}[c_\gamma]$, for some f such that $\pi([f]) = \gamma$. So the set $\{\alpha < \kappa : f(\alpha) \in \gamma\}$ is in $\mathcal{U}$, hence since $\mathcal{U}$ is κ-complete, f has constant value some $\beta < \gamma$ on a set in $\mathcal{U}$. But then $[f] = [c_\beta]$, and so $\gamma = \pi([f]) = \pi([c_\beta]) = j(\beta) = \beta$, which is impossible. This shows j is constant below κ. ☐

Measurable cardinals and elementary embeddings

Continued.

Now let *id* be the identity function on κ. Clearly, $[c_\beta]E_\mathcal{U}[id]E_\mathcal{U}[c_\kappa]$, for all $\beta < \kappa$. Thus, $\beta = j(\beta) < \pi([id]) < j(\kappa)$, for all $\beta < \kappa$. Hence, $\kappa < j(\kappa)$.

For the converse, suppose $j : V \to M$ is an elementary embedding, with M transitive, and with $\kappa = crit(j)$. Define $\mathcal{U}$ as follows:

$$X \in \mathcal{U} \text{ iff } X \subseteq \kappa \text{ and } \kappa \in j(X).$$

It is easy to see that $\mathcal{U}$ is an ultrafilter over κ. Notice that for every $\alpha < \kappa$, $j(\{\alpha\}) = \{\alpha\}$, and so $\mathcal{U}$ is non-principal. ☐

Measurable cardinals and elementary embeddings

Continued.

Let us check it is κ-complete. So let $\{X_\alpha : \alpha < \beta\} \subseteq \mathcal{U}$, some $\beta < \kappa$, and let $X := \bigcap_{\alpha<\beta} X_\alpha$. Then,

$$\kappa \in \bigcap_{\alpha<\beta} j(X_\alpha) = \bigcap_{\alpha<j(\beta)} j(X_\alpha) = j(\bigcap_{\alpha<\beta} X_\alpha) = j(X)$$

and so $X \in \mathcal{U}$. ☐

Normal ultrafilters

A filter on a regular uncountable cardinal is called normal if it is closed under diagonal intersections. We already saw that for every regular uncountable cardinal κ the $Cub(\kappa)$ filter is normal.

Every principal filter on κ is normal.

Let us observe that the ultrafilter $\mathcal{U}$ defined at the end of the last proof is normal. For suppose $\{X_\alpha : \alpha < \kappa\} \subseteq \mathcal{U}$. Recall that $\Delta_{\alpha<\kappa} X_\alpha$ is defined as the set $\{\alpha < \kappa : \alpha \in \bigcap_{\beta<\alpha} X_\beta\}$. So,

$$\kappa \in \{\alpha < j(\kappa) : \alpha \in \bigcap_{\beta<\alpha} X_\beta\} = j(\Delta_{\alpha<\kappa} X_\alpha)$$

and so $\Delta_{\alpha<\kappa} X_\alpha \in \mathcal{U}$.

Measurable cardinals and elementary embeddings

Theorem (Keisler and Scott, 1961)

κ is measurable if and only if there exists an elementary embedding $j : V \to M$, with M transitive, such that $\kappa = crit(j)$.

Measurable cardinals are weakly compact

Theorem

If κ is measurable, then κ is weakly compact.

Proof.

Fix a partition $f : [\kappa]^2 \to 2$. Let $\mathcal{U}$ be a κ-complete, non-principal, normal ultrafilter on κ. For each $\alpha < \kappa$, let $f_\alpha : [\kappa]^1 \to 2$ be given by: $f_\alpha(\beta) = f(\{\alpha, \beta\})$. Since $\mathcal{U}$ is an ultrafilter, for each $\alpha < \kappa$ there is $X_\alpha \in \mathcal{U}$ that is f_α-homogeneous, with constant value i_α.
Let $X := \Delta_{\alpha<\kappa} X_\alpha$. Since $\mathcal{U}$ is normal, $X \in \mathcal{U}$. If $\alpha, \beta \in X$ and $\alpha < \beta < \kappa$, then $\beta \in X_\alpha$, and so $f(\{\alpha, \beta\}) = i_\alpha$. Let $i \in \{0, 1\}$ and $H \subseteq X$, $H \in \mathcal{U}$, be such that $i_\alpha = i$ for all $\alpha \in H$. Then $f(\{\alpha, \beta\}) = i$ for all $\alpha, \beta \in H$. ☐

Measurable cardinals

Suppose $\mathcal{U}$ is a κ-complete non-principal ultrafilter on κ, and let $j : V \to M \cong Ult(V,\mathcal{U})$ be the corresponding ultrapower embedding. Then

1. $M^\kappa \subseteq M$.
2. $\mathcal{U} \notin M$
3. $2^\kappa < j(\kappa) < (2^\kappa)^+$

Note that (1) implies that $V_{\kappa+1} \subseteq M$, and (2) implies that $M \neq V$.

Measurable cardinals

If $\mathcal{U}$ is an ultrafilter on a regular uncountable cardinal κ, then $\mathcal{U}$ is normal if and only in for every regressive function f on a set $S \in \mathcal{U}$ there exists $S' \in \mathcal{U}$ contained in S on which f is constant.

If $\mathcal{U}$ is a κ-complete and normal non-principal ultrafilter over κ, then it contains all cub subsets of κ, and therefore every element of $\mathcal{U}$ is stationary.

Now suppose $\mathcal{U}$ is a normal κ-complete non-principal ultrafilter over κ. In $Ult(V,\mathcal{U})$, suppose $[f]E_{\mathcal{U}}[id]$. Then f is regressive on a set in $\mathcal{U}$. Hence, it is constant on a set in $\mathcal{U}$, and so $[f]=[c_\alpha]$, for some $\alpha<\kappa$.
Also, clearly $[c_\alpha]E_{\mathcal{U}}[id]$, for all $\alpha<\kappa$. Thus, we must have $\kappa=\pi([id])$.

Measurable cardinals

So suppose κ is measurable and $\mathcal{U}$ is a κ-complete non-principal ultrafilter on κ which is normal. Let $j : V \to M$ be the corresponding ultrapower embedding.
Since $V_{\kappa+1} \subseteq M$, and since κ is weakly compact in V, we have that κ is also weakly compact in M.
But since $\mathcal{U}$ is normal, $\kappa = \pi([id])$. Hence, in $Ult(V, \mathcal{U})$, $[id]$ is weakly compact.
It follows that the set of weakly compact cardinals smaller than κ belongs to $\mathcal{U}$, and so it is stationary.

Outline

4 Lecture IV

- Strongly compact cardinals
- Supercompact cardinals
- Extendible cardinals

Strongly compact cardinals

An uncountable cardinal κ is called strongly compact if for every set *I*, every κ-complete filter on *I* can be extended to a κ-complete ultrafilter on *I*.

Thus, since for κ regular the filter consisting on all subsets of κ whose complement has cardinality less than κ is κ-complete and non-principal, every strongly compact cardinal is measurable.

Definition

If $\delta \leq \kappa$ are uncountable cardinals, we say that κ is δ-strongly compact if for every set *I*, every κ-complete filter on *I* can be extended to a δ-complete ultrafilter on *I*. Thus, κ is strongly-compact iff it is κ-strongly compact.

Strongly compact cardinals

Notice that if κ is δ-strongly compact and λ is a cardinal greater than κ, then λ is also δ-strongly compact. Also note that if κ is regular and ω_1-strongly compact, then there exists a measurable cardinal less or equal than κ.

Suppose κ is δ-strongly compact. Let I be any non-empty set, and for every $a \in I$, let $X_a = \{x \in \mathcal{P}_\kappa(I) : a \in x\}$, where $\mathcal{P}_\kappa(I) = \{x \subseteq I : |x| < \kappa\}$. If κ is regular, then the set $\{X_a : a \in I\}$ generates a κ-complete filter on $\mathcal{P}_\kappa(I)$, which can be extended to a δ-complete ultrafilter on $\mathcal{P}_\kappa(I)$. Such an ultrafilter $\mathcal{U}$ is called a δ-complete fine measure on $\mathcal{P}_\kappa(I)$. The fineness condition is that $X_a \in \mathcal{U}$ for all $a \in I$.

Strongly compact cardinals

We have the following characterizations of δ-strong compactness.

Proposition

The following are equivalent for any uncountable cardinals $\delta \leq \kappa$:

1. κ *is* δ*-strongly compact.*
2. *For every* α *greater or equal than* κ *there exists an elementary embedding* $j : V \to M$*, with* M *transitive, and critical point greater or equal than* δ*, such that* j *is definable in* V*, and there exists* $D \in M$ *such that* $j''\alpha \subseteq D$ *and* $M \models |D| < j(\kappa)$*.*
3. *For every set* I *there exists a* δ*-complete fine measure on* $\mathcal{P}_\kappa(I)$*.*

Strongly compact cardinals

If λ is the least measurable cardinal and κ is ω_1-strongly compact, then κ is λ-strongly compact. For if $\mathcal{U}$ is a ω_1-complete ultrafilter on a set I that is not λ-complete, then there is a partition $\{X_\alpha : \alpha < \beta\}$ of I, some $\beta < \lambda$, such that none of the X_α belongs to $\mathcal{U}$. But then the set $\{X \subseteq \beta : \bigcup_{\alpha \in X} X_\alpha \in \mathcal{U}\}$ is a non-principal ω_1-complete ultrafilter on β, contradicting the minimality of λ.

Thus if κ is ω_1-strongly compact and is also the first measurable, a consistent situation as shown by Magidor, then κ is in fact strongly compact.

Supercompact cardinals

In the spirit of extending naturally the notion of measurable cardinal by requiring that M is close to V, we have the following notion of large cardinal:

Definition (Solovay, Reinhardt)

Let γ be an ordinal. A cardinal κ is γ-supercompact if there exists $j : V \to M$ with $c.p.(j) = \kappa$ and $M^\gamma \subseteq M$.

It can be shown that if κ is γ-supercompact, say witnessed by $j : V \to M$, then for some $n < \omega$, the nth-iterate of j, call it j^n, also witnesses the γ-supercompactness of κ and, moreover, $\gamma < j^n(\kappa)$. Thus, we may, and will, require in the defintion of γ-supercompactness that $\gamma < j(\kappa)$.

Supercompact cardinals

Thus, κ is measurable iff it is γ-supercompact for some (for all) $\gamma < \kappa^+$.
Suppose that κ is 2^κ-supercompact, witnessed by $j : V \to M$.
Let $\mathcal{U}$ be the ultrafilter derived from j, i.e.,

$$X \in \mathcal{U} \text{ iff } X \subseteq \kappa \text{ and } \kappa \in j(X).$$

Since $M^{2^\kappa} \subseteq M$, $\mathcal{U} \in M$. Hence, κ is measurable in M and, therefore, the set of measurable cardinals below κ belongs to $\mathcal{U}$.

If κ is supercompact, then $V_\kappa \prec_2 V$.

Supercompact cardinals

Definition

κ is supercompact if it is γ-supercompact for all γ.

We have defined the notion of γ-supercompactness only in terms of elementary embeddings of the universe into a transitive class. We want now to find, as in the case of measurable cardinals, an equivalent formulation in terms of the existence of some sets so that the corresponding elementary embeddings will be definable from those sets. In the case of a measurable cardinal κ, the sets were ultrafilters on κ. Now, we know the embeddings cannot come (for $\gamma \geq \kappa^+$) from ultrafilters on κ, but perhaps they may come from ultrafilters on some other set.

Supercompact cardinals

Proposition

Suppose $\mathcal{U}$ is a σ-complete ultrafilter over a set A and $j : V \to M$ is the corresponding elementary embedding. Then for every ordinal γ, $j''\gamma \in M$ iff $M^\gamma \subseteq M$.

Recall that if κ is measurable and $j : V \to M$ has critical point κ, then $j''\kappa = \kappa$ and $M^\kappa \subseteq M$. Then we defined the ultrafilter associated to j as the collection of all $X \subseteq \kappa$ such that $\kappa \in j(X)$. So, now suppose $j : V \to M$ witnesses the γ-supercompactness of κ. Since $M^\gamma \subseteq M$, it seems only natural to define an ultrafilter associated to j, call it $\mathcal{U}$, as:

$$X \in \mathcal{U} \text{ iff } X \subseteq \mathcal{P}_\kappa(\gamma) \text{ and } j''\gamma \in j(X).$$

Supercompact cardinals

The following can be easily checked:

Proposition

1. *$\mathcal{U}$ is a κ-complete ultrafilter on $\mathcal{P}_\kappa(\gamma)$.*
2. *$\mathcal{U}$ is fine, i.e., for every $\alpha < \gamma$, $\{X \in \mathcal{P}_\kappa(\gamma) : \alpha \in X\} \in \mathcal{U}$.*
3. *$\mathcal{U}$ is normal, i.e., if $\langle X_\alpha : \alpha < \gamma \rangle$ is a sequence of sets from $\mathcal{U}$, then its diagonal intersection $\Delta_{\alpha<\gamma} X_\alpha$ belongs to $\mathcal{U}$.*

Supercompact cardinals

Proof.

(1): First notice that since $\gamma < j(\kappa)$, $j''\gamma \in j(\mathcal{P}_\kappa(\gamma)) = (\mathcal{P}_{j(\kappa)}(j(\gamma)))^M$, and so $\mathcal{P}_\kappa(\gamma) \in \mathcal{U}$. The rest is straightforward.

(2): Need to check that $j''\gamma \in j(\{X \in \mathcal{P}_\kappa(\gamma) : \alpha \in X\}) = \{X \in \mathcal{P}_{j(\kappa)}(j(\gamma)) : j(\alpha) \in X\}$. But since $\gamma < j(\kappa)$, this is obvious.

(3): Need to check that $j''\gamma \in j(\{x \in \mathcal{P}_\kappa(\gamma) : x \in \bigcap_{\alpha \in x} X_\alpha\}) = \{x \in \mathcal{P}_{j(\kappa)}(j(\gamma)) : x \in \bigcap_{\alpha \in x} j(X_\alpha)\}$. Since $\gamma < j(\kappa)$, this is obvious. ☐

Supercompact cardinals

If $\mathcal{U}$ is as above and $j : V \to M$ is the associated elementary embedding, then $[id] = j''\gamma$. Hence, for every function f on $\mathcal{P}_\kappa(\gamma)$, $[f] = [c_f]([id]) = j_{\mathcal{U}}(f))(j''\gamma)$.

If $\mathcal{U}$ is a fine measure on $\mathcal{P}_\kappa(\gamma)$, then $\mathcal{U}$ is normal iff whenever $f : \mathcal{P}_\kappa(\gamma) \to V$ is such that $f(X) \in X$ for almost all X, then f is constant for almost all X. (Why?: for the same reason as for measures on a cardinal κ. Fineness plays the role in this case as the fact that, in the case of measures on κ, final segments have measure 1.)

Supercompact cardinals

Definition

A supercompact measure on $\mathcal{P}_\kappa(\gamma)$ is a κ-complete, fine and normal ultrafilter on $\mathcal{P}_\kappa(\gamma)$.

Theorem

If $\kappa \leq \gamma$, then κ is γ-supercompact iff there is a supercompact measure on $\mathcal{P}_\kappa(\gamma)$.

Supercompact cardinals

Proof.

We have just proved one direction, namely, if $j : V \to M$ witnesses the γ-supercompactness of κ, then $\mathcal{U} = \{X \subseteq \mathcal{P}_\kappa(\gamma) : j''\gamma \in j(X)\}$ is a supercompact measure. Conversely, if $\mathcal{U}$ is a supercompact measure on $\mathcal{P}_\kappa(\gamma)$, let $j : V \to M$ be the associated elementary embedding. Let us first check that $j(\kappa) > \gamma$. Let f be the function that assigns to every element of $\mathcal{P}_\kappa(\gamma)$ its order type, i.e., $f(X) = o.t.(X)$. We have that $[f] = j_{\mathcal{U}}(f)(j''\gamma) = o.t.(j''\gamma) = \gamma$. Hence, since $o.t.(X) < \kappa$ for all $X \in \mathcal{P}_\kappa(\gamma)$, we have $\gamma < j(\kappa)$. ☐

Supercompact cardinals

Continued.

To see that $M^\gamma \subseteq M$ it is enough to show that $j''\gamma \in M$. For each $\alpha < \gamma$, let $f_\alpha : \mathcal{P}_\kappa(\gamma) \to OR$ be such that $j(\alpha) = [f_\alpha]$. Let now f be the function with domain $\mathcal{P}_\kappa(\gamma)$ given by:
$f(X) = \{f_\alpha(X) : \alpha \in X\}$. We claim that $[f] = j''\gamma$.
By fineness of $\mathcal{U}$, for every $\alpha < \gamma$, $\alpha \in X$ for almost all $X \in \mathcal{P}_\kappa(\gamma)$. Hence, for almost all X, $f_\alpha(X) \in f(X)$, and so $[f_\alpha] \in [f]$. On the other hand, if $[g] \in [f]$, then $g(X) \in f(X)$ for almost all X, and so for almost all X, $g(X) = f_\alpha(X)$ for some $\alpha \in X$. By normality applied to the function $g'(X) =$ the α such that $g(X) = f_\alpha(X)$, there is $\alpha < \gamma$ such that $g(X) = f_\alpha(X)$ for almost all X, and so $[g] = [f_\alpha] = j(\alpha)$. □

Strongly compact and supecompact cardinals

Thus, if κ is supercompact, then it is strongly compact. However, the converse is not true.

Theorem (Magidor, 1976)

1. *If κ is supercompact, then there is a forcing extension in which κ is supercompact and is also the least strongly compact cardinal.*
2. *If κ is strongly compact, then there is a forcing extension in which it is still strongly compact and is also the first measurable cardinal.*

The strongest large cardinals

κ is called a Reinhardt cardinal if there exists an elementary embedding $j : V \to V$ with critical point κ.

Theorem (Kunen, 1971)

Reinhardt cardinals don't exist.

In fact, Kunen proves that there doesn't exist any non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$.

The existence of an elementary embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ is one of the strongest large cardinal principles not known to be inconsistent.

Extendible cardinals

A cardinal κ is λ–extendible if there is an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with critical point κ and such that $j(\kappa) > \lambda$. And κ is extendible if it is λ–extendible for all $\lambda > \kappa$.

The next lemma implies that every extendible cardinal is supercompact.

Lemma (Magidor)

Suppose $j : V_\lambda \to V_\mu$ is elementary, λ is a limit ordinal, and κ is the critical point of j. Then κ is $< \lambda$-supercompact.

Extendible cardinals

Proof.

Fix $\gamma < \lambda$ and define

$$\mathcal{U}_\gamma = \{X \subseteq \mathcal{P}_\kappa(\gamma) : j''\gamma \in j(X)\}.$$

Note that this makes sense if $j(\kappa) > \gamma$, in which case it is easy to check that $\mathcal{U}_\gamma$ is a κ-complete, fine, and normal measure. Otherwise, let $j^1 = j$ and $j^{m+1} = j \circ j^m$. If $j^m(\kappa) > \gamma$ for some m, then define $\mathcal{U}_\gamma$ using j^m instead of j. But such an m does exist, for otherwise $\delta := sup_m(j^m(\kappa)) \leq \gamma < \lambda$, and then since $j(\delta) = \delta$ we would have $j \restriction V_{\delta+2} : V_{\delta+2} \to V_{\delta+2}$ is elementary with critical point κ, contradicting Kunen's Theorem. ☐

Extendible cardinals

If κ is extendible, then the set of supercompact cardinals smaller than κ is stationary.

Outline

5 Lecture V
- Vopenka’s Principle
- Accessible categories

Vopěnka's Principle

Definition (Vopěnka's Principle (VP). P. Vopěnka, 1960's)

There is no rigid proper class of graphs. I.e., no proper class of graphs in which the only morphisms are the identity morphisms.

Equivalently, for every proper class $\mathcal{C}$ of structures of the same type, there exist $A \neq B$ in $\mathcal{C}$ such that A is (elementarily) embeddable into B.

Vopenka's Principle

VP can be formulated in the first-order language of set theory as an axiom schema, i.e., as an infinite set of axioms, one for each formula with two free variables. Formally, for each such formula $\varphi(x, y)$ one has the axiom:

$$\forall x[(\forall y \forall z(\varphi(x, y) \wedge \varphi(x, z) \rightarrow y \text{ and } z \text{ are structures of same type}) \wedge$$

$$\forall \alpha \in OR\ \exists y(rank(y) > \alpha \wedge \varphi(x, y)) \rightarrow$$

$$\exists y \exists z(\varphi(x, y) \wedge \varphi(x, z) \wedge y \neq z \wedge \exists e(e : y \rightarrow z \text{ is elementary}))].$$

The theory ZFC plus VP implies, for instance, that the class of extendible cardinals is stationary, i.e., every definable cub proper class contains an extendible cardinal.

The H_κ

For an infinite cardinal κ, H_κ is the set of all sets having transitive closure of cardinality $< \kappa$. Thus, $H_\omega = V_\omega$. We always have $H_\kappa \subseteq V_\kappa$. But $H_{\omega_1} \neq V_{\omega_1}$, as e.g., $\mathcal{P}(\omega) \in V_{\omega+2} \setminus H_{\omega_1}$. Note that all H_κ are transitive.

Similarly as with the V_α, the H_κ also form a cumulative hierarchy: if $\kappa \leq \lambda$, then $H_\kappa \subseteq H_\lambda$, and if κ is a limit cardinal, then $H_\kappa = \bigcup_{\lambda<\kappa} H_\lambda$. Finally, $V = \bigcup_{\kappa \in CARD} H_\kappa$.

There is a closed proper class of cardinals C such that $V_\kappa = H_\kappa$, for every $\kappa \in C$.

If κ is inaccessible, then $V_\kappa = H_\kappa$.

Variants of VP

Let us consider the following variants of VP, the first one apparently much stronger than the second.
We say that a class $\mathcal{C}$ is Σ_n (Π_n) if it is definable, with parameters, by a Σ_n (Π_n) formula of the language of set theory.

Definition

If Γ is one of Σ_n, Π_n, some $n \in \omega$, and κ is an infinite cardinal, then we write $VP(\kappa, \Gamma)$ for the following assertion:

For every Γ proper class $\mathcal{C}$ of structures of the same type τ such that both τ and the parameters of some Γ-definition of $\mathcal{C}$, if any, belong to H_κ, $\mathcal{C}$ reflects below κ, i.e., for every $B \in \mathcal{C}$, there exists $A \in \mathcal{C} \cap H_\kappa$ that is elementarily embeddable into B.

Variants of VP

Definition

If Γ is one of Σ_n, Π_n, some $n \in \omega$, we write $VP(\Gamma)$ for the following statement:

For every Γ proper class $\mathcal{C}$ of structures of the language of set theory with one (equivalently, finitely-many) additional 1-ary relation symbol(s), there exist distinct A and B in $\mathcal{C}$ with an elementary embedding of A into B.

Vopenka's Principle

VP for Σ_1 classes is a consequence of ZFC. In fact, the following holds.

Theorem

If κ is an uncountable cardinal, then every (not necessarily proper) class $\mathcal{C}$ of structures of the same type $\tau \in H_\kappa$ which is Σ_1 definable, with parameters in H_κ, reflects below κ. Hence, VP(κ, Σ_1) holds for every uncountable cardinal κ.

Vopenka's Principle

Proof.

Fix an uncountable cardinal κ and a class $\mathcal{C}$ of structures of the same type $\tau \in H_\kappa$, definable by a Σ_1 formula with parameters in H_κ.
Given $B \in \mathcal{C}$, let λ be a regular cardinal greater than κ, with $B \in H_\lambda$, and let N be an elementary substructure of H_λ, of cardinality less than κ, which contains B and the transitive closure of $\{\tau\}$ together with the parameters involved in some Σ_1 definition of $\mathcal{C}$.
Let A and M be the transitive collapses of B and N, respectively, and let $j : M \to N$ be the collapsing isomorphism.
Then $A \in H_\kappa$, and $j \restriction A : A \to B$ is an elementary embedding.
Observe that $j(\tau) = \tau$. So, since Σ_1 formulas are upwards absolute for transitive models, and since $M \models A \in \mathcal{C}$, we have that $A \in \mathcal{C}$. ☐

Vopenka's Principle

In contrast, Vopěnka's Principle for Π_1 proper classes implies the existence of very large cardinals.

Theorem

If $VP(\Pi_1)$ holds, then there exists a supercompact cardinal.

Vopenka's Principle

Proof.

Let $\mathcal{C}$ be the class of structures of the form $\langle V_{\lambda+2}, \in, \alpha, \lambda\rangle$, where λ is the least limit ordinal greater than α such that no $\kappa \leq \alpha$ is $< \lambda$-supercompact.

We claim that $\mathcal{C}$ is Π_1 definable without parameters. For $X \in \mathcal{C}$ if and only if $X = \langle X_0, X_1, X_2, X_3\rangle$, where

(1) X_2 is an ordinal

(2) X_3 is a limit ordinal greater than X_2

(3) $X_0 = V_{X_3+2}$

(4) $X_1 = \in\restriction X_0$

(5) And the following hold in $\langle X_0, X_1\rangle$:

1. $\forall \kappa \leq X_2(\kappa$ is not $< X_3$-supercompact$)$
2. $\forall \mu(\mu$ limit $\wedge\ X_2 < \mu < X_3 \rightarrow \exists \kappa \leq X_2(\kappa$ is $<$ μ-supercompact$))$.

Vopenka's Principle

Continued.

If there is no supercompact cardinal, then $\mathcal{C}$ is a proper class. So by $VP(\Pi_1)$, there exist $\langle V_{\lambda+2}, \in, \alpha, \lambda\rangle \neq \langle V_{\mu+2}, \in, \beta, \mu\rangle$ and an elementary embedding

$$j : \langle V_{\lambda+2}, \in, \alpha, \lambda\rangle \to \langle V_{\mu+2}, \in, \beta, \mu\rangle.$$

Since j must send α to β and λ to μ, j is not the identity. Hence by Kunen's theorem we must have $\lambda < \mu$, and therefore also $\alpha < \beta$. So, j has critical point some $\kappa \leq \alpha$. It now follows from Magidor's Lemma that κ is $< \lambda$-supercompact. But this is impossible because $\langle V_{\lambda+2}, \in, \alpha, \lambda\rangle \in \mathcal{C}$. ☐

Vopenka's Principle

We give next a strong converse to the last Theorem.

Theorem (BCMR)

Suppose that $\mathcal{C}$ *is a* Σ_2 *(not necessarily proper) class of structures of the same type* τ*, and suppose that there exists a supercompact cardinal* κ *larger than the rank of the parameters that appear in some* Σ_2 *definition of* $\mathcal{C}$*, and with* $\tau \in V_\kappa$*. Then for every* $B \in \mathcal{C}$ *there exists* $A \in \mathcal{C} \cap V_\kappa$ *that is elementarily embeddable into* B*.*

Proof.

Fix a Σ_2 formula $\varphi(x, y)$ and a set b such that $\mathcal{C} = \{B : \varphi(B, b)\}$, and suppose that κ is a supercompact cardinal with $b \in V_\kappa$. Fix $B \in \mathcal{C}$, and let $\lambda \in C^{(2)}$ be greater than rank(B). □

Vopenka's Principle

Continued.

Let $j\colon V \to M$ be an elementary embedding with M transitive and critical point κ, such that $j(\kappa) > \lambda$ and M is closed under λ-sequences. Thus, B and $j \restriction B : B \to j(B)$ are in M, and also $V_\lambda \in M$. Hence $V_\lambda \preceq_1 M$. Moreover, since $j(\tau) = \tau$, $j(B)$ is a tructure of type τ, and $j \restriction B$ is an elementary embedding. Since $V_\lambda \preceq_2 V$, $V_\lambda \models \varphi(B, b)$. And since Σ_2 formulas are upwards absolute between V_λ and M, $M \models \varphi(B, b)$. Thus, in M it is true that there exists $X \in M_{j(\kappa)}$ such that $\varphi(X, b)$, namely B, and there exists an elementary embedding $e\colon X \to j(B)$, namely $j \restriction B$. Therefore, by elementarity, the same holds in V; that is, there exists $X \in V_\kappa$ such that $\varphi(X, b)$, and there exists an elementary embedding $e\colon X \to B$. $\square$

Vopenka's Principle

The following corollaries give characterizations of Vopěnka's principle for Π_1 and Σ_2 classes in terms of supercompactness.

Corollary

The following are equivalent:

1. *VP*(Π_1).
2. *VP*(κ, Σ_2), *for some* κ.
3. *There exists a supercompact cardinal.*

An observation

Given a Σ_{n+1} definable class of structures $\mathcal{C}$, say via the Σ_{n+1} formula $\varphi(x)$, let $\mathcal{C}^*$ be the class of structures of the form $A^* = \langle V_\alpha, \in, A\rangle$, where α is the least ordinal in $C^{(n)}$ such that $V_\alpha \models \varphi(A)$. Then,

$$A \in \mathcal{C} \text{ if and only if } A^* \in \mathcal{C}^*.$$

Then $\mathcal{C}^*$ is Π_n definable. This explains why, e.g., $VP(\Pi_n)$ is equivalent to $VP(\Sigma_{n+1})$, or why a cardinal reflects Π_n classes if and only if it reflects Σ_{n+1} classes.

Vopěnka's Principle

We shall give next a characterization of supercompactness in terms of a natural principle of reflection.

Recall that a cardinal κ reflects a class of structures $\mathcal{C}$ of the same type if for every $B \in \mathcal{C}$ there exists $A \in \mathcal{C} \cap H_\kappa$ which is elementary embeddable into B.

Theorem (Magidor)

If κ is the least cardinal that reflects the Π_1 proper class $\mathcal{C}$ of structures of the form $\langle V_\lambda, \in \rangle$, then κ is supercompact.

Supercompactness as a reflection principle

The last two theorems yield the following characterizations of the first supercompact cardinal.

Corollary

The following are equivalent:

1. κ *is the first supercompact cardinal.*
2. κ *is the least ordinal that reflects all* Σ_2 *definable, with parameters in* V_κ*, classes of structures of the same type. i.e.,* κ *is the least ordinal for which* $VP(\kappa, \Sigma_2)$ *holds.*
3. κ *is the least ordinal that reflects the* Π_1 *class of structures of the form* $\langle V_\lambda, \in\rangle$*,* λ *an ordinal.*

Corollary

A cardinal κ *reflects all* Π_1 *classes of structures of the same type iff either* κ *is supercompact or a limit of supercompacts.*

$C^{(n)}$–extendible cardinals

We will see next similar results for classes of higher complexity, for which we shall need $C^{(n)}$-extendible cardinals.

Definition (BCMR, 2008; B, 2010)

κ is $C^{(n)}$-extendible if for every λ greater than κ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with $crit(j) = \kappa$, $j(\kappa) > \lambda$, and $V_{j(\kappa)}$ is a Σ_n-elementary substructure of V.

$C^{(n)}$–extendible cardinals

Theorem (BCMR, 2008; B, 2010)

For every $n \geq 1$, if κ is a $C^{(n)}$–extendible cardinal, then every class $\mathcal{C}$ of structures of the same type $\tau \in H_\kappa$ which is Σ_{n+2} definable, with parameters in H_κ, reflects below κ. Hence $VP(\kappa, \Sigma_{n+2})$ holds.

$C^{(n)}$–extendible cardinals

The next theorem yields a strong converse to Theorem above.

Theorem (BCMR, 2008; B, 2010)

Suppose $n \geq 1$*. If* $VP(\Pi_{n+1})$ *holds, then there exists a* $C^{(n)}$*-extendible cardinal.*

The following corollaries summarizes the results above.

Corollary (BCMR, 2008; B, 2010)

The following are equivalent:

1. $VP(\Pi_2)$.
2. $VP(\kappa, \Sigma_3)$*, for some* κ.
3. *There exists an extendible cardinal.*

$C^{(n)}$–extendible cardinals

Corollary (BCMR, 2008; B, 2010)

The following are equivalent for $n \geq 1$:

1. $VP(\Pi_{n+1})$.
2. $VP(\kappa, \Sigma_{n+2})$, *for some* κ.
3. *There exists a* $C^{(n)}$*-extendible cardinal.*

Corollary (BCMR, 2008; B, 2010)

The following are equivalent:

1. $VP(\Pi_n)$, *for every* n.
2. $VP(\kappa, \Sigma_n)$, *for a proper class of cardinals* κ, *and for every* n.
3. VP
4. *For every* n, *there exists a* $C^{(n)}$*-extendible cardinal.*

$C^{(n)}$-extendible cardinals and reflection

We give next a characterization of $C^{(n)}$-extendible cardinals in terms of reflection of classes of structures.

Theorem (B, 2010)

Suppose $n \geq 1$ and κ is the least cardinal that reflects all Π_{n+1} proper classes of structures of the same type, then κ is $C^{(n)}$-extendible.

$C^{(n)}$-extendible cardinals and reflection

Corollary (BCMR, 2008; B, 2010)

The following are equivalent for each $n \geq 1$*:*

1. κ *is the least* $C^{(n)}$*-extendible cardinal.*
2. κ *is the least ordinal that reflects all* Σ_{n+2} *definable, with parameters in* V_κ*, classes of structures of the same type. i.e.,* κ *is the least ordinal for which* $VP(\kappa, \Sigma_{n+2})$ *holds.*
3. κ *is the least cardinal that reflects all* Π_{n+1} *proper classes of structures of type* $\langle V_\alpha, \in, A\rangle$*, where* A *is a unary predicate.*

$C^{(n)}$-extendible cardinals and reflection

The following parameterized version also follows.

Theorem (B, 2010)

A cardinal κ reflects all Π_{n+1} (proper) classes of structures of the same type if and only if either κ is a $C^{(n)}$-extendible cardinal or a limit of $C^{(n)}$-extendible cardinals.

Definable categories

A category $\mathcal{C}$ is definable with a set of parameters p if there are formulas

$$\psi_{\mathrm{Ob}}(x,p), \quad \psi_{\mathrm{Mor}}(x,y,z,p), \quad \psi_{\circ}(x_1,\dots,x_6,p), \quad \psi_{id}(x,y,p) \tag{2}$$

such that:

1. A is an object of $\mathcal{C}$ if and only if $\psi_{\mathrm{Ob}}(A,p)$ is true.
2. The sentence $\psi_{\mathrm{Mor}}(A,B,f,p)$ is true if and only if $f \in \mathcal{C}(A,B)$.
3. The sentence $\psi_{\circ}(A,B,C,f,g,h,p)$ is true if and only if $f \in \mathcal{C}(A,B)$, $g \in \mathcal{C}(B,C)$, $h \in \mathcal{C}(A,C)$, and h is the composite of f and g.
4. The sentence $\psi_{id}(A,i,p)$ is true if and only if A is an object of $\mathcal{C}$ and i is the identity of A.

Categories of structures

If T is a theory in a (possibly infinitary) language $\mathcal{L}$, then the category **Mod**(T) whose objects are the models of T and whose maps are the structure homomorphisms is definable. If the formulas of T are finitary, then **Mod**(T) is defined by a bounded formula (i.e., Σ_0) with T and $\mathcal{L}$ as parameters.

Every accessible category is equivalent to **Mod**(T), for some basic theory T. I.e., every formula in T is of the form $\forall x(\varphi(x) \to \psi(x))$, where $\varphi(x)$ and $\psi(x)$ are positive-existential formulas.

Thus, accessible categories definable in a finitary language are Σ_0 definable, with parameters.

A more complex example is the homotopy category of simplicial sets, which is Σ_2 definable.

Definable categories

Theorem (BCMR, 2010)

For an uncountable cardinal κ and a language $\mathcal{L}$, let $\mathcal{C}$ be a full subcategory of $\mathcal{L}$-structures defined by a Σ_1 formula with a set of parameters p. Suppose that p and $\mathcal{L}$ are in $H(\kappa)$. Then every object $B \in \mathcal{C}$ has a subobject $A \in \mathcal{C} \cap H(\kappa)$.

Theorem (BCMR, 2010)

For every supercompact cardinal κ and every Σ_2 subcategory $\mathcal{C}$ of sets defined with parameters of rank less than κ and supporting elementary embeddings with critical point κ, every object $B \in \mathcal{C}$ has a subobject $A \in \mathcal{C} \cap V_\kappa$.

Weakly-preaccessible categories

Let λ be a regular cardinal. A category $\mathcal{D}$ is called λ-filtered if, given any set of objects $\{A_i\}_{i\in I}$ where $|I| < \lambda$, there is an object A and a morphism $A_i \to A$ for each $i \in I$, and, moreover, given any set of parallel arrows between any two objects $\{f_j \colon B \to C\}_{j\in J}$ where $|J| < \lambda$, there is a morphism $g \colon C \to D$ such that $g \circ f_j$ is the same morphism for all $j \in J$.

If $\mathcal{C}$ is any category, a functor $F \colon \mathcal{D} \to \mathcal{C}$ where $\mathcal{D}$ is a λ-filtered small category is called a λ-filtered diagram.

For each subcategory $\mathcal{D}$ of $\mathcal{C}$ and every object A in $\mathcal{C}$, the canonical diagram sends each morphism $X \to A$ to X.

Weakly-preaccessible categories

A full subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is called dense if each object *A* of $\mathcal{C}$ is a colimit of the canonical diagram.
A category $\mathcal{C}$ is bounded if it has a small dense full subcategory.
If $\mathcal{C}$ has a small dense full subcategory $\mathcal{D}$ such that the canonical diagram is λ-filtered for every object *A* of $\mathcal{C}$, then we say that $\mathcal{C}$ is weakly λ-preaccessible.
A category $\mathcal{C}$ will be called weakly preaccessible if it is weakly λ-preaccessible for some regular cardinal λ.

Accessible categories

Theorem (BCMR, 2010)

Let $\mathcal{C}$ be an accessible category. Then:

(1) *Every Σ_1 full subcategory of $\mathcal{C}$ is weakly preaccessible.*

(2) *If there is a proper class of supercompact cardinals, then every Σ_2 full subcategory of $\mathcal{C}$ is weakly preaccessible.*

(3) *For $n \geq 1$, if there is a proper class of $C^{(n)}$-extendible cardinals, then every Σ_{n+2} full subcategory of $\mathcal{C}$ is weakly preaccessible.*

Corollary

If there is a proper class of supercompact cardinals, then every accessible category is co-wellpowered.

Accessible categories

Theorem (BCMR, 2010)

Assume the existence of a proper class of $C^{(n)}$*-extendible cardinals, where* $n \geq 1$. *Then each* Σ_{n+1} *orthogonality class in an accessible category is a small-orthogonality class.*