Set-Theoretic Aspects of Abelian Groups

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Preliminaries

Set theory is ZFC.

In these talks, groups are abelian. Equivalently, they are $\mathbb{Z}$-modules.

In a group, *(linear) independence* means over $\mathbb{Z}$.

An (abelian) group G is *free* if it has a *basis*, i.e., an independent generating set.

A homomorphism from a free G to any other group H is determined by its values at a basis, and these values are arbitrary.

If (G_i) is a family of free groups, linearly ordered by "subgroup of", with bases (X_i), **and** if the X_i are also linearly ordered by inclusion, then $\bigcup_i G_i$ is free with basis $\bigcup_i X_i$.

Free groups are *projective*, i.e., if F is free, then any epimorphism $f : G \to F$ with kernel K is, up to isomorphism, the projection $F \oplus K \to F$.

It follows that, if K is also free, then so is G, and each basis for K is included in some basis for G.

Subgroups of Free Groups Are Free

Suppose G is a subgroup of a free group F, and B is a basis for F.

Fix a well-ordering $<$ of B.

For each $b \in B$, let F_b be the subgroup generated by the predecessors of b in B.

By induction on b, show that $G \cap F_b$ is free, **and** construct a basis X_b such that $X_{b'} \subseteq X_b$ whenever $b' < b$.

That makes the induction easy at limit steps.

Induction step for successors:

Let b be the immediate successor of c.

The projection $F_b \to \mathbb{Z}$ with kernel F_c restricts to a homomorphism $G \cap F_b \to \mathbb{Z}$ with free kernel $G \cap F_c$ and free image, a subgroup of $\mathbb{Z}$.

How Not to Be Free

Torsion prevents freeness.

If G has an element $x \neq 0$ such that $nx = 0$ for some positive integer n, then G is not free.

From now on, work with *torsion-free* groups.

A subgroup $G \subseteq H$ is *pure* in H if H/G is torsion-free.
Equivalently: If $x \in H$ and $nx \in G$ for an integer $n \neq 0$, then $x \in G$.
The *purification* of G in H is the smallest pure subgroup of H that includes G.

Divisibility prevents freeness.

Example: $\mathbb{Q}$.
More generally, if G has a finitely generated subgroup whose purification is not finitely generated, then G is not free.

These are the only **algebraic** obstructions to freeness.

Other failures of freeness are essentially **set-theoretic**.

Pontryagin's Criterion

If

- G is a torsion-free abelian group,
- every finitely generated subgroup of G is included in a finitely generated pure subgroup of G, and
- G is countable,

then G is free.

Without the countability hypothesis, we still get that all countable subgroups of G are free, but G itself might not be free.

The Baer-Specker Group

Notation: $\Pi = \mathbb{Z}^\omega$ is the group of infinite sequences of integers (with componentwise addition).

Σ is the subgroup of Π consisting of sequences with finite support.

Σ is free, with basis $\{\mathbf{e}_i : i \in \omega\}$, where $\mathbf{e}_i$ has a single non-0 component, a 1 in position i (a "standard unit vector").

All countable subgroups of Π are free, by Pontryagin's criterion.

But Π is not free.

Lemma: If F is free of uncountable cardinality and $G \subseteq F$ is a subgroup of cardinality κ, then F/G is the direct sum of a free group and a group of cardinality $\leq \kappa$.

Proof: Fix a basis X for F. Only a subset $X_0 \subseteq X$ of size $\leq \kappa$ is involved in the X-expressions for elements of G. Let F_0 and F_1 be the subgroups of F freely generated by X_0 and $X - X_0$. So $F = F_0 \oplus F_1$ and $G \subseteq F_0$. Therefore, $F/G = (F_0/G) \oplus F_1$.

Π is not free

Corollary: Under the lemma's hypotheses, a divisible subgroup of F/G has size at most κ.

But the divisible part of Π/Σ has cardinality $\mathfrak{c} = 2^{\aleph_0}$, whereas $|\Sigma| = \aleph_0$. So Π is not free.

In fact, Π is very far from free.

A free group of size $\mathfrak{c}$ would have a basis of size $\mathfrak{c}$ and therefore

$$2^{\mathfrak{c}} = 2^{2^{\aleph_0}}$$

homomorphisms to $\mathbb{Z}$.

But Π has only $\aleph_0$ homomorphisms to $\mathbb{Z}$.

Proposition (Specker): A homomorphism $h : \Pi \to \mathbb{Z}$ is determined by $h \restriction \Sigma$ (equivalently by the $h(\mathbf{e}_i)$'s for $i \in \omega$).

Proof: Let D_2 be the subgroup of Π consisting of those sequences (x_i) such that, for each k, all sufficiently large i have $2^k | x_i$.

In D_2/Σ, every element is divisible by 2. So D_2/Σ has no non-zero homomorphisms to $\mathbb{Z}$. So $h \restriction \Sigma$ determines $h \restriction D_2$.

Similarly, $h \restriction \Sigma$ determines $h \restriction D_3$.

But D_2 and D_3 together generate Π.

The Specker Phenomenon

Proposition (Specker): For any homomorphism $h : \Pi \to \mathbb{Z}$, there are only finitely many $i \in \omega$ with $h(\mathbf{e}_i) \neq 0$.

Proof: Suppose h is a counterexample; $A = \{i \in \omega : h(\mathbf{e}_i) \neq 0\}$ is infinite. Define $(x_i) \in \Pi$ by setting $x_i = 0$ for $i \notin A$ and, for $i \in A$, choosing x_i and distinct odd primes p_i inductively as follows.

$p_i \not| h(\mathbf{e}_i)$.

$p_j | x_i$ for all $j < i$.

$\sum_{j \leq i} x_j h(\mathbf{e}_j) \equiv \frac{p_i - 1}{2} \pmod{p_i}$.

Then, for each $i \in A$,

$$h((x_k)_{k \in \omega}) =$$

$$\Big(\sum_{j \leq i} x_j h(\mathbf{e}_j)\Big) + h(p_i(0, \ldots, 0, \frac{x_{i+1}}{p_i}, \frac{x_{i+2}}{p_i}, \ldots)$$

$$\equiv \frac{p_i - 1}{2} \pmod{p_i}.$$

But no integer satisfies all these congruences.

Reflexivity

Reformulation of the preceding results:
The dual $\Pi^* = \mathrm{Hom}(\Pi, \mathbb{Z})$ of Π is Σ (via the obvious "inner product" pairing).

Easily, the dual of Σ is Π (via the same inner product pairing).

So the canonical homomorphism from Σ to its double dual

$$x \mapsto (f \mapsto f(x))$$

is an isomorphism.
One says Σ is *reflexive*.
Π is also reflexive.
Are all free groups reflexive?
Theorem: For any cardinal κ, the following are equivalent.

- The free group on κ generators is reflexive.
- Homomorphisms $\mathbb{Z}^\kappa \to \mathbb{Z}$ are determined by their values on the standard unit vectors $\mathbf{e}_\alpha$ (for $\alpha < \kappa$).
- Every countably complete ultrafilter on κ is principal.
- No measurable cardinal is $\leq \kappa$.

Łoś-Eda Theorem

If $\mathcal{U}$ is a non-principal, countably complete ultrafilter on κ, then there is a non-zero homomorphism $h : \mathbb{Z}^\kappa \to \mathbb{Z}$ that sends all $\mathbf{e}_\alpha$ to 0.

Let h send any $(x_\alpha)_{\alpha<\kappa}$ to the common value of $x_\alpha \in \mathbb{Z}$ for $\mathcal{U}$-almost all α.

Conversely, from a non-zero homomorphism $h : \mathbb{Z}^\kappa \to \mathbb{Z}$ that sends all $\mathbf{e}_\alpha$ to 0, one can construct a non-principal, countably complete ultrafilter on κ. In fact, any such h is a finite $\mathbb{Z}$-linear combination of homomorphisms constructed as above from such ultrafilters.

From Homomorphism to Ultrafilter

Let $h : \mathbb{Z}^\kappa \to \mathbb{Z}$ be non-zero while $h(\mathbf{e}_\alpha) = 0$ for all $\alpha < \kappa$.

Define

$$\begin{aligned} \mathcal{U} = \{X \subseteq \kappa : (\exists \mathbf{x} = (x_\alpha)_{\alpha\in\kappa} \in \Pi) \\ \big(h(\mathbf{x}) \neq 0 \wedge (\forall \alpha \notin X)\, x_\alpha = 0\big)\}. \end{aligned}$$

Claim 1: $\mathcal{U}$ does not contain infinitely many pairwise disjoint sets.

Proof: If disjoint sets X_n, for $n \in \omega$, are in $\mathcal{U}$, witnessed by vectors $\mathbf{x}^n$, then there is a well-defined homomorphism

$$q : \mathbb{Z}^\omega \to \mathbb{Z}^\kappa : (y_n)_{n\in\omega} \mapsto \sum_n y_n \mathbf{x}^n.$$

The composition $h \circ q : \mathbb{Z}^\omega \to \mathbb{Z}$ sends each $\mathbf{e}_i$ to

$$h(q(\mathbf{e}_i)) = h(\mathbf{x}^i) \neq 0,$$

a contradiction to one of the Specker propositions.

Ultrafilter, continued

Claim 2: There is an $R \in \mathcal{U}$ such that no two disjoint subsets of R are in $\mathcal{U}$.

Otherwise, we could repeatedly split sets in $\mathcal{U}$ to contradict Claim 1.

Fix such an R, and let $\mathcal{U}' = \mathcal{U} \cap \mathcal{P}(R)$.

Claim 3: If R is split into a (finite or countable) number of pieces, then one of the pieces is in $\mathcal{U}'$.

Proof for the countably infinite case: Let the pieces be A_n, and let $\mathbf{x}$ witness that $R \in \mathcal{U}$. Let $(\mathbf{x} \restriction A_n)$ agree with $\mathbf{x}$ on A_n and be 0 outside A_n. Then

$$q : \mathbb{Z}^\omega \to \mathbb{Z}^\kappa : (y_n)_{n\in\omega} \mapsto \sum_n y_n \cdot (\mathbf{x} \restriction A_n)$$

sends the all-ones sequence to $\mathbf{x}$. So $h \circ q : \mathbb{Z}^\omega \to \mathbb{Z}$ is not identically 0. So there is $i \in \omega$ with

$$0 \neq h(q(\mathbf{e}_i)) = h(x \restriction A_i).$$

So $A_i \in \mathcal{U}'$.

The case of finite partitions is easier.

Ultrafilter, continued

Claim 4: If $A, B \in \mathcal{U}'$ then $A \cap B \in \mathcal{U}'$.

Proof: By Claim 3, one of $A \cap B$, $A - B$, $B - A$, and $R - A - B$ is in $\mathcal{U}'$. If it were any of the last three, that would contradict the choice of R.

The claims (and the obvious fact that $\mathcal{U}'$ is closed under supersets) imply that $\mathcal{U}'$ is a countably complete ultrafilter on R.

Two elements of $\mathbb{Z}^\kappa$ supported on R and agreeing on a set in $\mathcal{U}'$ have the same image under h.

Combine with Claim 3 to get that h has a non-0 value at some $\mathbf{x}$ that is supported on R and constant on R.

So h has a non-zero value v on the characteristic function of R.

So, for any $\mathbf{x}$ supported on R, $h(\mathbf{x})$ is v times the $\mathcal{U}'$-almost everywhere value of $\mathbf{x}$.

There could be other, disjoint such R's, each with its own $\mathcal{U}'$, but only finitely many, by Claim 1.

Filtrations

Any group G of uncountable cardinality admits a *filtration*, a sequence, $\langle G_\xi : \xi < \lambda \rangle$, such that

- the length λ is $\mathrm{cf}(|G|)$,
- each G_ξ is a pure subgroup of G,
- $G_\xi \subseteq G_\eta$ whenever $\xi < \eta$,
- $G_\eta = \bigcup_{\xi<\eta} G_\xi$ for all limit ordinals $\eta < \lambda$,
- $G = \bigcup_{\xi<\lambda} G_\xi$, and
- $|G_\xi| < |G|$ for all $\xi < \lambda$.

If $\langle G_\xi : \xi < \lambda \rangle$ and $\langle G'_\xi : \xi < \lambda \rangle$ are two filtrations of the same G and if $|G|$ is an uncountable regular cardinal, then $G_\xi = G'_\xi$ for a closed unbounded (in $|G|$) set of ξ's.

If $|G| = \kappa^+$ then we can arrange for all subgroups in a filtration of G to have cardinality κ.

If G is the supremum of an increasing, continuous sequence of cardinals $\langle \kappa_\xi : \xi < \lambda \rangle$ then we can arrange for a filtration to satisfy $|G_\xi| = \kappa_\xi$.

Filtrations and Freeness

The following are equivalent, for any group of uncountable cardinality:

1. G is free.
2. For some filtration $\langle G_\xi : \xi < \lambda \rangle$ of G, G_0 and all the quotients $G_{\xi+1}/G_\xi$ are free.
3. For some filtration $\langle G_\xi : \xi < \lambda \rangle$ of G, G_0 and all the quotients G_η/G_ξ, for $\xi < \eta$, are free.

For regular $|G| > \omega$, also equivalent is:

For every filtration $\langle G_\xi : \xi < \lambda \rangle$ of G, there is a club of ξ's such that, for all $\eta > \xi$, the quotients G_η/G_ξ are free.

Proof Sketch: To go from (1) to (3), fix a basis X for G and a "filtration" $\langle X_\xi : \xi < \lambda \rangle$ of X. Let G_ξ be generated by X_ξ.

To go from (2) to (1), construct, by induction on ξ, bases X_ξ for G_ξ such that $X_\xi \subseteq X_\eta$ for $\xi < \eta$.

To handle *all* filtrations when $\mathrm{cf}(G)$ is uncountable, use the facts that any two filtrations agree on a club and that subgroups of free groups are free.

Gamma Invariants

Let G be a group of uncountable, regular cardinality κ, such that all subgroups of smaller cardinality are free (we say G is *κ-free*).
Let $\langle G_\xi : \xi < \kappa \rangle$ be a filtration of G.
The *Gamma invariant* $\Gamma(G)$ of G is the equivalence class, modulo the club filter on κ, of

$$\{\xi < \kappa : G/G_\xi \text{ is not } \kappa\text{-free}\}.$$

This set depends on the filtration, but the equivalence class $\Gamma(G)$ does not.

Because κ is regular, every subset of G of size $< \kappa$ is $\subseteq G_\xi$ for some ξ.
So G/G_ξ is κ-free if and only if G_η/G_ξ is free for every $\eta > \xi$.

G is free if and only if $\Gamma(G) = [\varnothing]$.

If G is not free, the failure of freeness is measured by the element $\Gamma(G)$ of the Boolean algebra $\mathcal{P}(\kappa)/\text{club}$.

Forcing Freeness

If we enlarge the universe of sets by forcing, a group G that is not free might become free in the larger universe. One of the new sets might be a basis.

A group satisfies Pontryagin's criterion if and only if it becomes free in some forcing extension of the universe.

Let G be an $\aleph_1$-free group of size $\aleph_1$.

- If $\Gamma(G) = [\varnothing]$ then G is free.
- If $[\varnothing] < \Gamma(G) < [\aleph_1]$ then G is not free but it becomes free in a forcing extension that adds no new reals.
- If $\Gamma(G) = [\aleph_1]$ then the only way to force G to be free is to collapse $\aleph_1$.

If both G and G' are as above, and if their Gamma invariants are not $[\aleph_1]$ but together cover $[\aleph_1]$, then each one can be given a basis without adding reals, but the only way to make both free is to collapse $\aleph_1$.

If $\mathfrak{c} = \aleph_1$, so $\Gamma(\Pi)$ is defined, then $\Gamma(\Pi) = [\aleph_1]$. (Π is far from free.)

Compactness — Large Cardinals

If the cardinality of a group G is measurable and if all subgroups of smaller cardinality are free, then G is free.

Proof: Without loss of generality, the underlying set of G is a measurable cardinal κ.

Let $j : V \to M$ be a non-trivial elementary embedding of the universe V into a transitive class M, such that the first ordinal moved is κ.

Since V satisfies "All subgroups of G of size $< \kappa$ are free," elementarity says that M satisfies "All subgroups of $j(G)$ of size $< j(\kappa)$ are free." But G itself is such a subgroup because

- its underlying set is κ which is in M,
- its group operation agrees with that of $j(G)$ by elementarity, and
- $\kappa < j(\kappa)$.

So M sees a free basis X for G. Because M is a transitive class, the same X is a basis for G in V.

Compactness — Not Quite So Large Cardinals

The preceding result remains true if the hypothesis “measurable” is reduced to “weakly compact”.

Of the many equivalent formulations of weak compactness, the best for this proof is Π^1_1-indescribability.

If G is a group with underlying set κ, then the statement “G is not free” is Π^1_1 over V_κ.

Compactness — Singular Cardinals

Theorem (Shelah): If the cardinality of a group G is singular and if all subgroups of smaller cardinality are free, then G is free.

The proof uses the notion of *strongly κ-free* group, for regular uncountable κ.
This means a group with a *coherent system* $\mathcal{C}$ of free subgroups of size $< \kappa$.
Coherence means that $(0) \in \mathcal{C}$ and, if $C \in \mathcal{C}$ and A is any subset of G of size $< \kappa$, then there is $C' \in \mathcal{C}$ with $C \cup A \subseteq C'$ and C'/C free.

All the groups in a coherent system for G are pure subgroups of G.

Because subgroups of free groups are free, every strongly κ-free group is κ-free.

There is a weak converse, "sacrificing one cardinal".

Finding a Coherent System

Theorem: If κ is a regular, uncountable cardinal then every κ^+-free group is strongly κ-free.

Proof: Let G be κ^+-free.

To build a coherent system (of pure free subgroups of size $< \kappa$), begin with the system $\mathcal{F}$ of *all* such subgroups, and disqualify subgroups as necessary.

A group $C \in \mathcal{F}$ has *expiration date* α (an ordinal) if

- it does not have a smaller expiration date, and
- there is a *witness*, a subset A of G of size $< \kappa$, such that every $C' \in \mathcal{F}$ with $C \cup A \subseteq C'$ and C'/C free has expiration date $< \alpha$.

As $\mathcal{F}$ is only a set, eventually no new expiration dates are assigned. If (0) has no expiration date, then all the unexpired groups constitute a coherent system.

So suppose, toward a contradiction, that (0) has expiration date α.

Coherent System, continued

Every group $\in \mathcal{F}$ expired at or before α, because (0)'s witness also would serve as a witness for any of them that hadn't expired earlier.

Choose a witness $w(H)$ for each $H \in \mathcal{F}$.

Build an increasing, continuous κ-sequence $\langle H_\xi : \xi < \kappa \rangle$ of groups from $\mathcal{F}$ such that, for each ξ,

$$H_\xi \cup w(H_\xi) \subseteq H_{\xi+1}.$$

If, for some $\xi < \eta < \kappa$, it happens that H_η / H_ξ is free, then the expiration date of H_η is smaller than that of H_ξ.

The sequence $\langle H_\xi : \xi < \kappa \rangle$ is a filtration of its union.

The union is free, as G is κ^+-free.

So there is a club Q in κ such that, for $\xi < \eta$ both in Q, the quotient H_η / H_ξ is free.

So in this club, expiration dates strictly decrease — a contradiction.

Singular Compactness

Proof of Singular Compactness Theorem:

Assume λ is a singular cardinal, of cofinality $\mathrm{cf}(\lambda) = \mu < \lambda$.

Assume G is a λ-free group of size λ.

Fix a continuous, increasing μ-sequence of cardinals $\langle \kappa_\xi : \xi < \mu \rangle$, with supremum λ, and with $\kappa_0 > \mu$.

Fix a filtration $\langle G_\xi : \xi < \mu \rangle$ of G with $|G_\xi| = \kappa_\xi$ for all ξ.

For each ξ, since $\kappa_\xi^{++} < \lambda$, G is κ_ξ^{++}-free and thus strongly κ_ξ^{+}-free.

So fix, for each ξ, a coherent system $\mathcal{C}_\xi$ of free subgroups of G of size κ_ξ (except for (0)).

Singular Compactness, continued

Build a $\mu \times \omega$ matrix of subgroups $\langle M_{\xi,n} : \xi < \mu, n \in \omega \rangle$, with $M_{\xi,n} \in \mathcal{C}_\xi$, by the following induction on n (for all ξ simultaneously). Along with each $M_{\xi,n}$ we shall also choose a basis $X_{\xi,n}$ for it and a well-ordering of $X_{\xi,n}$ in order-type κ_ξ.

For $n = 0$, take $M_{\xi,0}$ to be any group in $\mathcal{C}_\xi$ that includes G_ξ; take $X_{\xi,0}$ to be any basis for it, necessarily of size κ_ξ; and well-order it abitrarily with order-type κ_ξ.

For $n + 1$, use, for each ξ, the fact that $M_{\xi,n}$ is in the coherent system $\mathcal{C}_\xi$ to choose $M_{\xi,n+1} \in \mathcal{C}_\xi$ so that:

1. $M_{\xi,n+1} \supseteq M_{\xi,n}$ with free quotient.
2. For all $\eta < \mu$, the first κ_ξ elements of $X_{\eta,n}$ are in $M_{\xi,n+1}$. (Remember that $\mu < \kappa_0 \leq \kappa_\xi$.)
3. All elements of $X_{\xi+1,n}$ occurring in the expansions of elements of $M_{\xi+1,n} \cap M_{\xi,n}$ are in $M_{\xi,n+1}$.

Singular Compactness, continued

Choose a basis $X_{\xi,n+1}$ for $M_{\xi,n+1}$ that includes the basis $X_{\xi,n}$ for $M_{\xi,n}$. Well-order it in order-type κ_ξ.

For each $\xi < \mu$, let $U_\xi = \bigcup_{n\in\omega} M_{\xi,n}$.
It has $Y_\xi = \bigcup_{n\in\omega} X_{\xi,n}$ as a free basis.
The sequence $\langle U_\xi : \xi < \mu\rangle$ is increasing. (If $\eta < \xi < \mu$ then each element of Y_η is in U_ξ by (2).)
When elements of U_ξ are expressed in terms of $Y_{\xi+1}$, all the elements of $Y_{\xi+1}$ that occur are in U_ξ by (3). So $U_{\xi+1}/U_\xi$ is free.
$\langle U_\xi : \xi < \mu\rangle$ is continuous at limit ordinals ξ. (If $a \in U_\xi$, express it in terms of basis elements from $X_{\xi,n}$ for some n. Those finitely many basis elements are among the first κ_η for some $\eta < \xi$ because the sequence of κ's was continuous. So by (2) these basis elements and therefore x are in $M_{\eta,n+1} \subseteq U_\eta$.)
Therefore, G, as the union of the filtration $\langle U_\xi : \xi < \mu\rangle$, is free.