

Large cardinals and colimits

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Preliminaries

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Let Σ be a signature, and let T be a theory in the language $\mathcal{L}_\alpha(\Sigma)$ for some cardinal α . We will be considering

- ▶ $\mathbf{Str}\Sigma$, the category of Σ -structures, and
- ▶ $\mathbf{Mod}T$, the category of models of T , thought of as a subcategory of $\mathbf{Str}\Sigma$,

under the assumption that there exists a certain elementary embedding $j : V \rightarrow M$ (more specifics on the large cardinal assumption later).

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First point to note:

- ▶ The inclusion $\mathbf{Mod}T \hookrightarrow \mathbf{Str}\Sigma$ is *full*: homomorphisms of T -models are simply the homomorphisms as Σ -structures.

For any set-sized structure A (in V),

A is a Σ -structure

is definable in the language of set theory, as is

$$A \models \varphi(a_1, \dots, a_n)$$

for any Σ -formula φ and elements $a_1, \dots, a_n \in A$. Moreover, the definition is absolute, so we can be sloppy about which universe we're in!

On the other hand, Σ is a parameter, and T is a parameter for $A \models T$, that is, $A \in \mathbf{Mod}T$. So account for this we'll only consider j with critical point sufficiently large that Σ and T are fixed by j .

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$$A \models \varphi(a_1, \dots, a_n) \text{ if and only if } j(A) \models \varphi(j(a_1), \dots, j(a_n)).$$

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- ▶ Further, $(j \upharpoonright \cdot)$ is a natural transformation from the identity functor to the functor j .

Proof of the last claim

The diagram whose commutativity we need to check is

$$\begin{array}{ccc} A & \xrightarrow{j \upharpoonright A} & j(A) \\ f \downarrow & & \downarrow j(f) \\ B & \xrightarrow{j \upharpoonright B} & j(B). \end{array}$$

For any $a \in A$, we indeed have

$$j(f) \circ j \upharpoonright A(a) = j(f)(j(a)) = j(f(a)) = j \upharpoonright B \circ f(a),$$

with the middle equality following from elementarity. □

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Theorem (Rosický)

Let α be an uncountable cardinal, T a theory for the language $\mathcal{L}_\alpha(\Sigma)$ over α -ary signature Σ , and suppose there exists a cardinal κ that is α -strongly compact. Then the inclusion $\mathbf{Mod} T \rightarrow \mathbf{Str} \Sigma$ preserves κ -directed colimits.

Warning!

It is not claimed that κ -directed colimits in **Mod** T exist (even though all colimits in **Str** Σ exist). Rather, the claim is that if a colimit in **Mod** T does exist, then it must be the colimit as calculated in **Str** Σ .

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Example

Let $\Sigma = \{\leq\}$, and let T be the theory of total orders with a maximum element (note: we don't have a constant symbol in Σ to represent this element, we just have $\exists x \forall y (x \geq y)$ as one of the sentences of T).

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Let \mathcal{D} be the diagram with finite ordinals as the objects (each with the usual order on it), and the usual inclusions as the morphisms. Then ω is the colimit in $\mathbf{Str} \Sigma$, but there is no colimit in $\mathbf{Mod} T$.

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(Why not $\omega + 1$? Because of the uniqueness requirement for colimits: for example, there are two order-preserving functions from $\omega + 1$ to $\omega + 2$ that respect the inclusions of finite ordinals, ie, fix ω .)

An embedding proof of Rosický's theorem

Let \mathcal{D} be a κ -directed diagram of models of T , with colimit B in the category $\mathbf{Mod}T$, and colimit A in the category $\mathbf{Str}A$; there is thus an induced homomorphism h from A to B . We shall show that A is in fact a model of T , and so h is an isomorphism.

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Recall that κ is α -strongly compact if and only if for every $\lambda > \kappa$ there is an elementary embedding $j : V \rightarrow M$ with critical point at least α , such that
there is a Z in M with
 $Z \supseteq j^{\lambda}$ and $M \models |Z| < j(\kappa)$.

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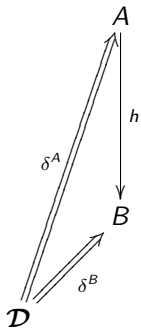
Recall that κ is α -strongly compact if and only if for every $\lambda > \kappa$ there is an elementary embedding $j : V \rightarrow M$ with critical point at least α , such that for every subset Y of M with $|Y| \leq \lambda$, there is a Z in M with $Z \supseteq j''Y$ and $M \models |Z| < j(\kappa)$.

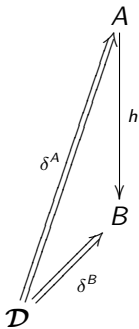
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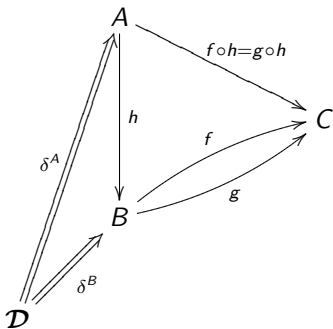
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Let us take such an embedding j for $\lambda = |\mathcal{D}|$.





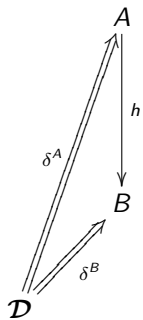
Note first that h is “epi for morphisms to T -models”:

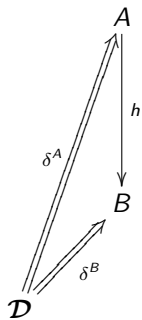


Note first that h is “epi for morphisms to T -models”: if f and g are homomorphisms from B to a T -model C such that $f \circ h = g \circ h$, then

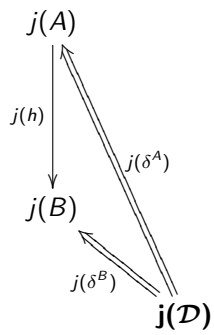
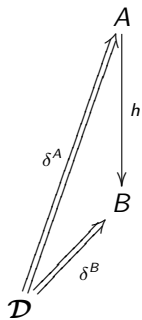
$$f \circ \delta^B = f \circ h \circ \delta^A = g \circ h \circ \delta^A = g \circ \delta^B,$$

so $f = g$ by the uniqueness of factorisation of a cocone through the colimit B .

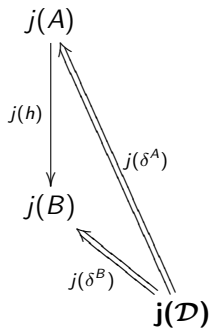
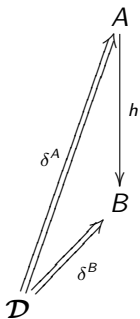




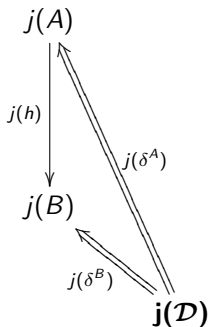
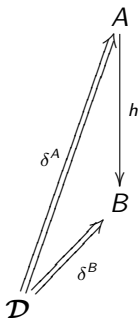
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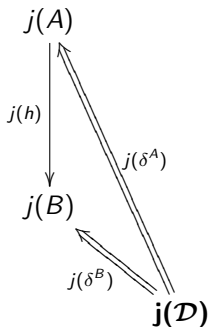
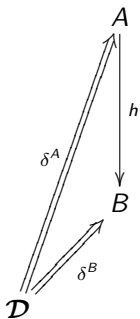


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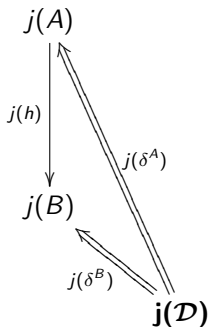
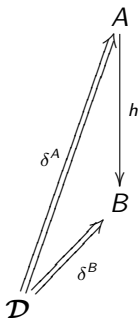
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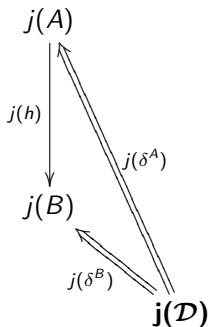
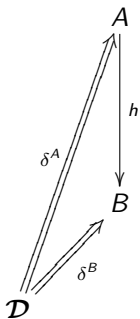
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- ▶ $j(B)$ is its colimit in **Mod** T ,



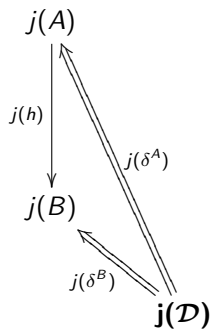
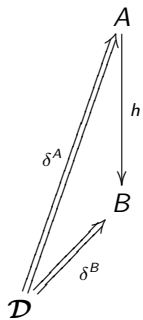
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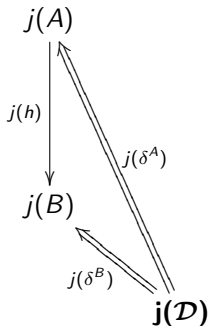
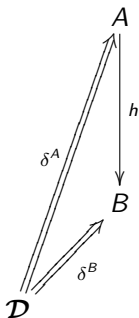
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- ▶ $j(A)$ is its colimit in $\mathbf{Str} \Sigma$, and



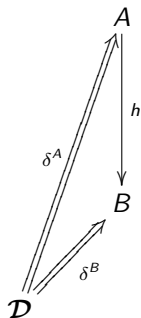
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- ▶ $j(h)$ is the induced homomorphism.

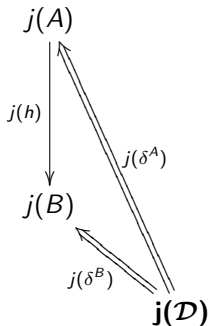




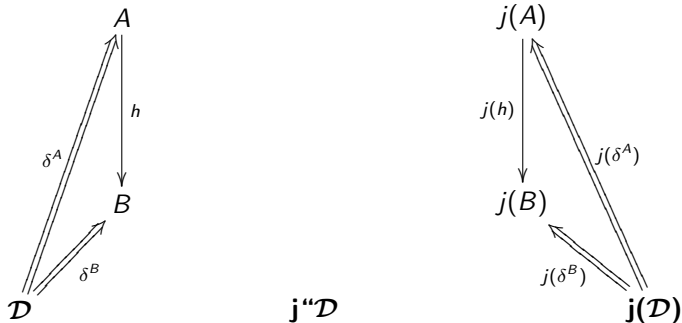
Consider $\mathbf{j}(\mathcal{D})$, with objects $\{j(D) \mid D \in \text{Obj}(\mathcal{D})\}$ and morphisms $\{j(f) \mid f \in \text{Mor}(\mathcal{D})\}$.



$j^*\mathcal{D}$

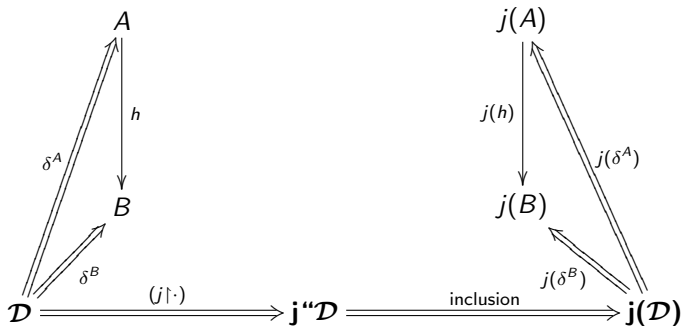


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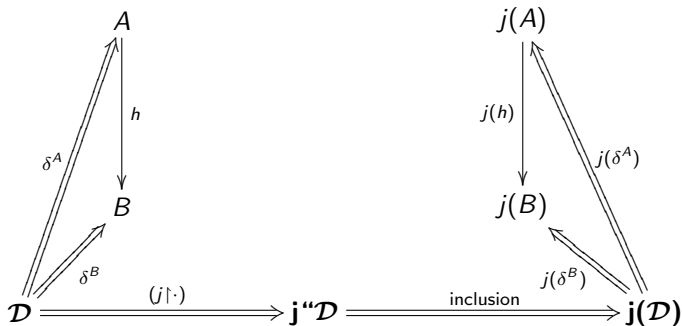
Consider $j^{\mathcal{D}}$, with objects $\{j(D) \mid D \in \text{Obj}(\mathcal{D})\}$ and morphisms $\{j(f) \mid f \in \text{Mor}(\mathcal{D})\}$.

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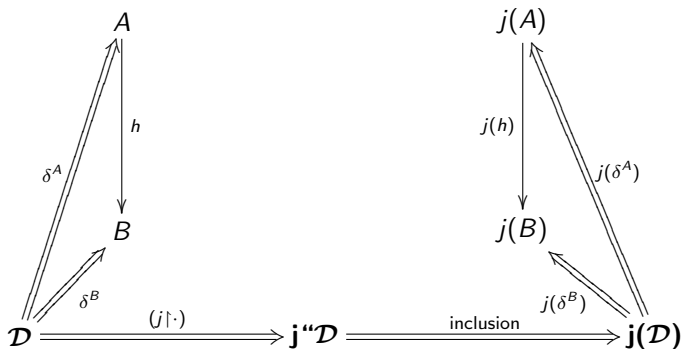
Consider $j|_{\mathcal{D}}$, with objects $\{j(D) \mid D \in \text{Obj}(\mathcal{D})\}$ and morphisms $\{j(f) \mid f \in \text{Mor}(\mathcal{D})\}$.

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Consider $j \uparrow \mathcal{D}$, with objects $\{j(D) \mid D \in \text{Obj}(\mathcal{D})\}$ and morphisms $\{j(f) \mid f \in \text{Mor}(\mathcal{D})\}$.

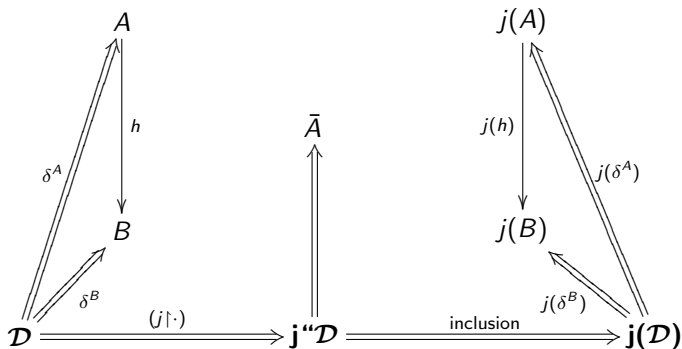
- ▶ It is the “image” of j considered as a functor restricted to \mathcal{D} ; it is a subdiagram of $j(\mathcal{D})$.
- ▶ It need not be an element of M , but it has cardinality $|j \uparrow \mathcal{D}| = \lambda$, so by the choice of j , there is a $Z \in M$ with $j \uparrow \text{Obj}(\mathcal{D}) \subseteq Z$ and $M \models |Z| < j(\kappa)$.



Now $Z \cap j(\mathcal{D})$ is a subset of \mathcal{D} of size less than $j(\kappa)$, so since

$M \models j(\mathcal{D})$ is $j(\kappa)$ -complete,

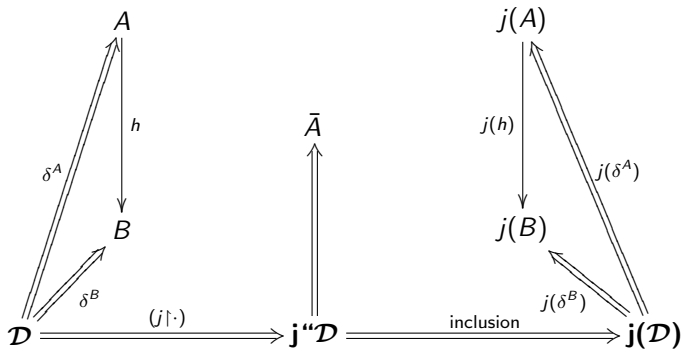
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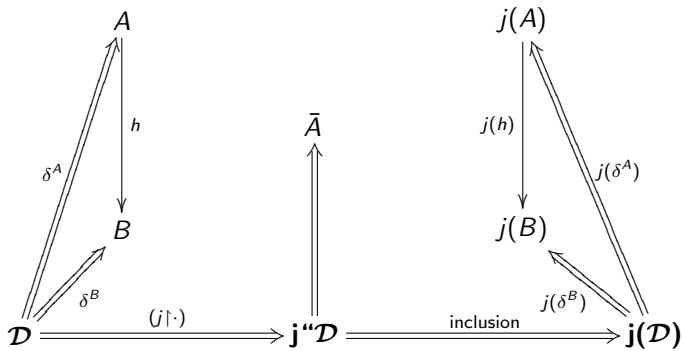
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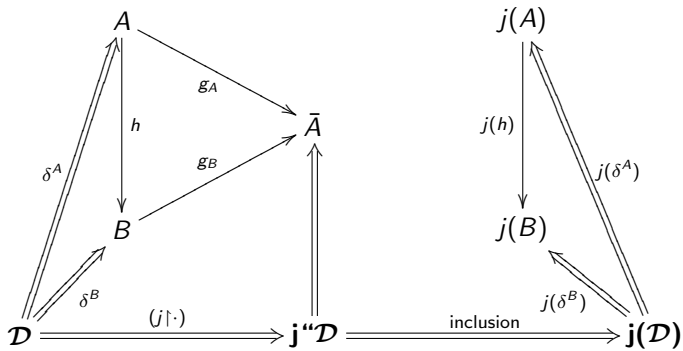
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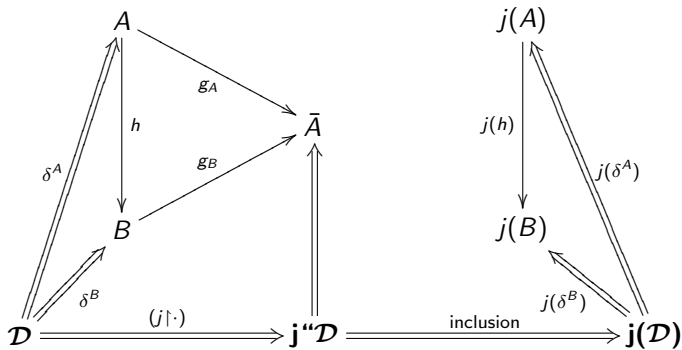
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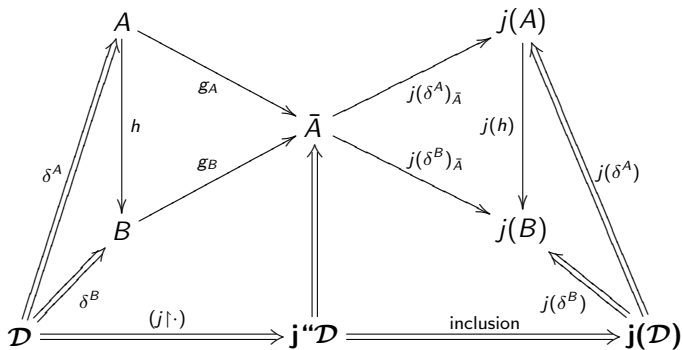
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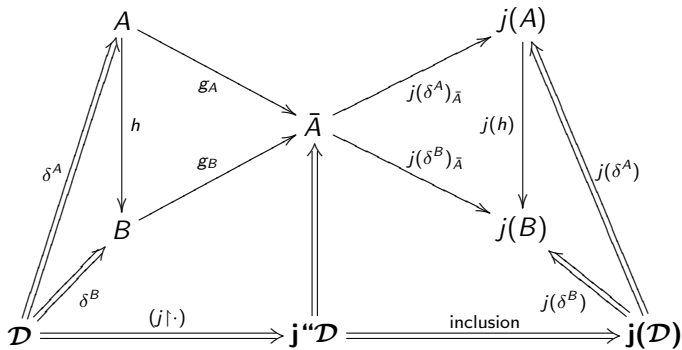
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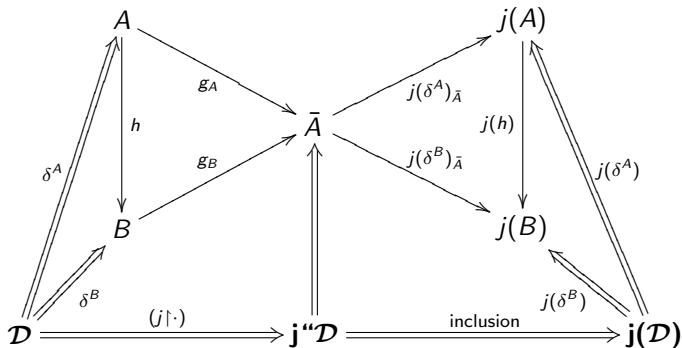
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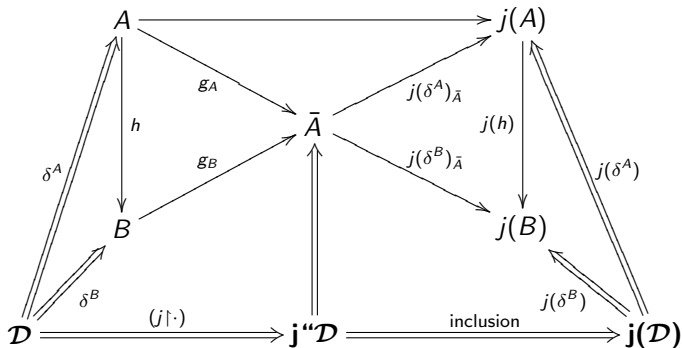
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Note: this whole diagram commutes!

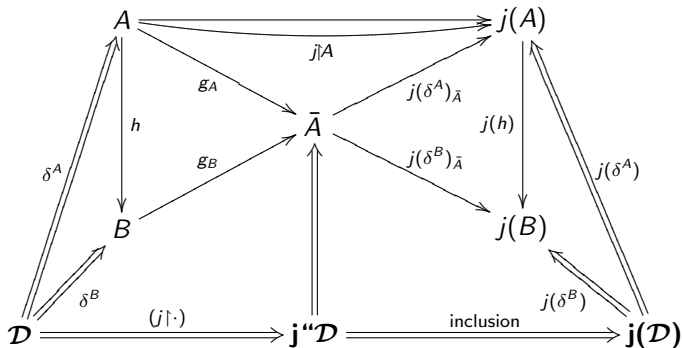


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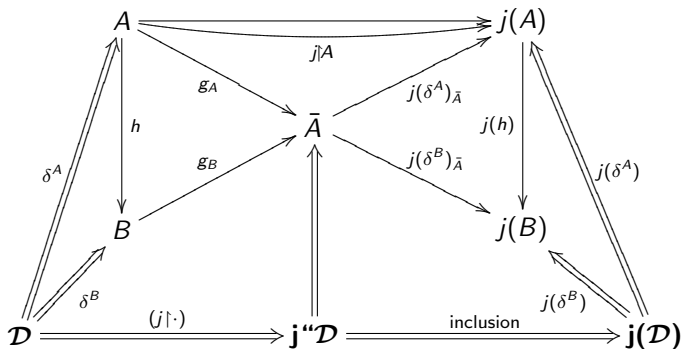
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- ▶ the map induced from the cone we have from \mathcal{D} to $j(A)$ since A is the colimit (which commutes with the diagram so far),



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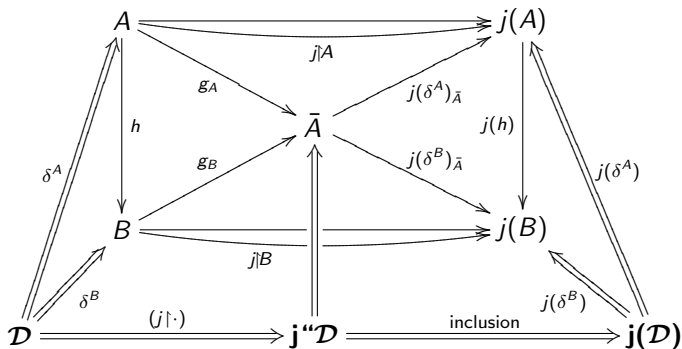
- ▶ the map induced from the cone we have from \mathcal{D} to $j(A)$ since A is the colimit (which commutes with the diagram so far), and
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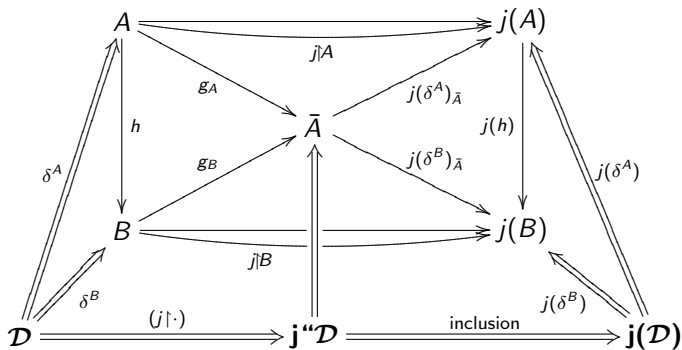
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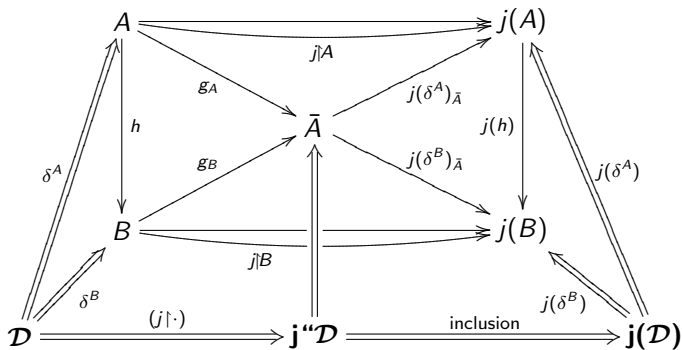
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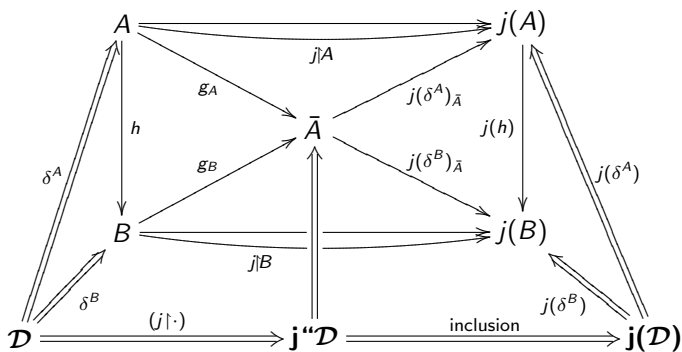
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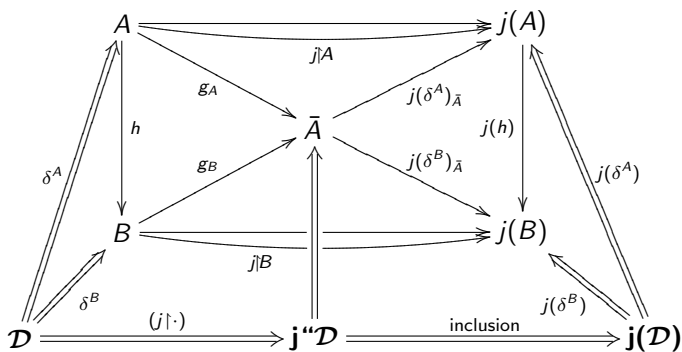
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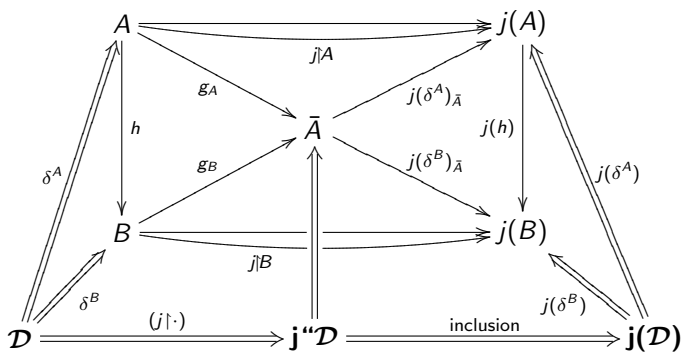
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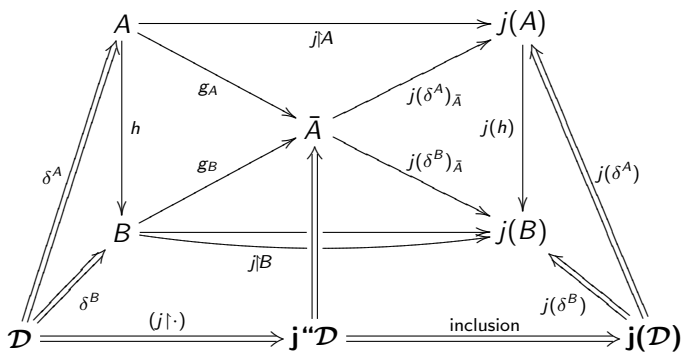
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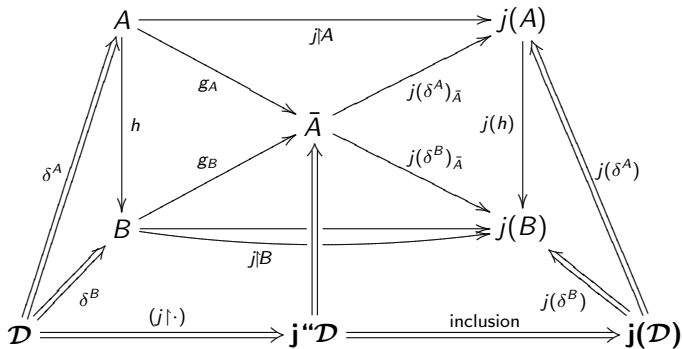
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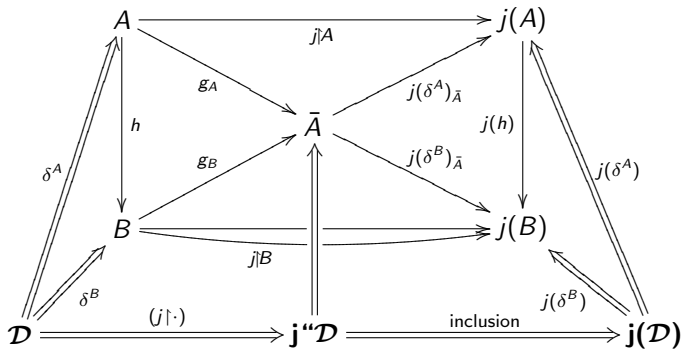
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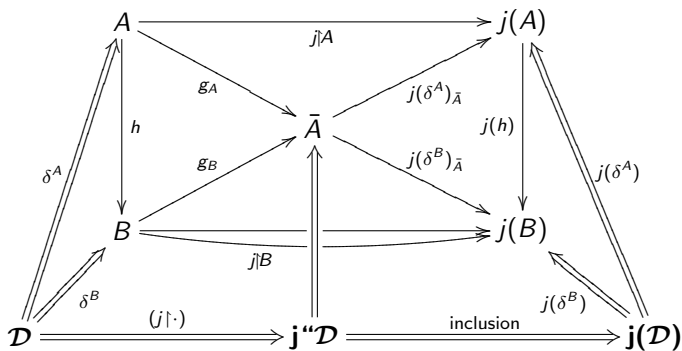
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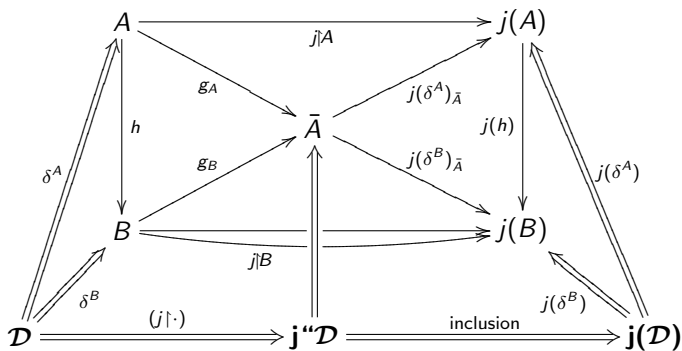
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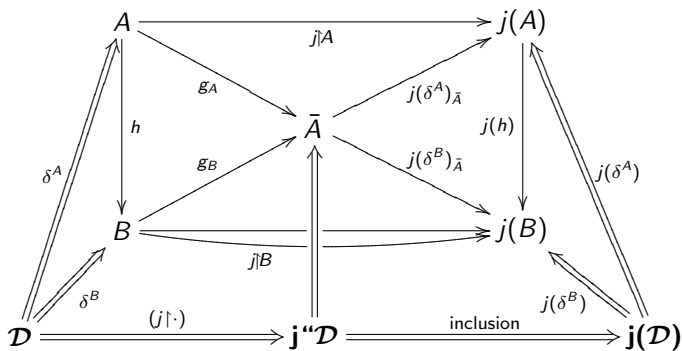
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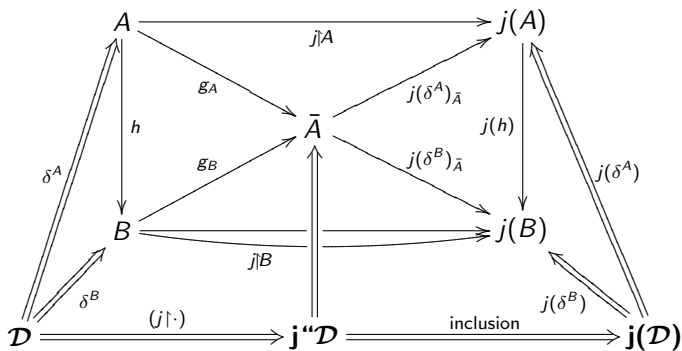
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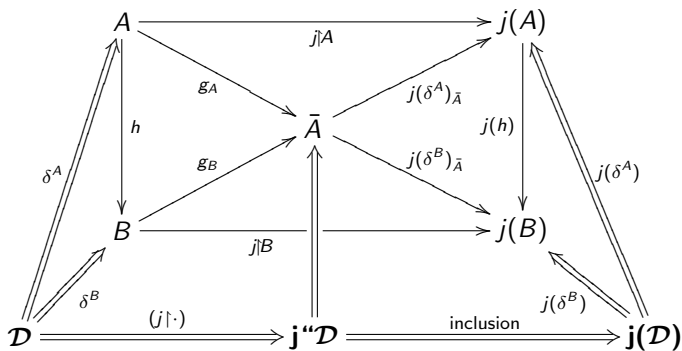
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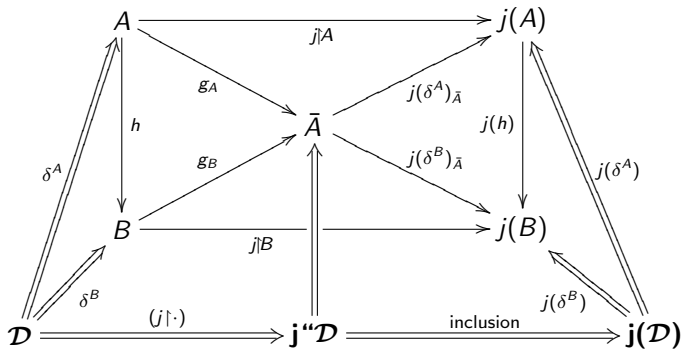
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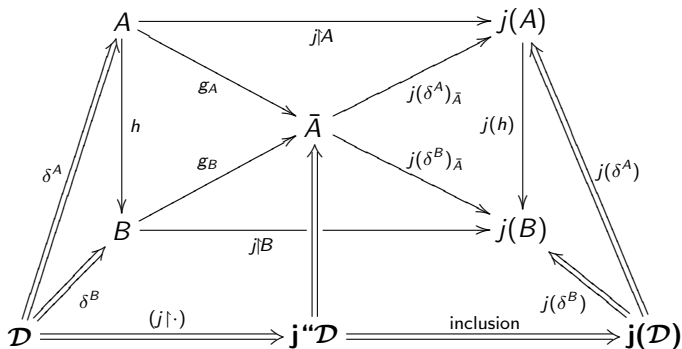


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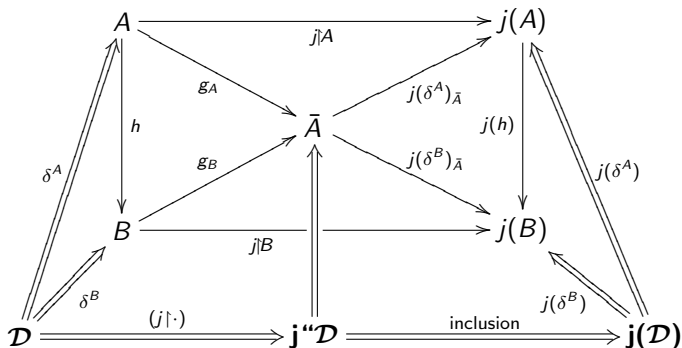


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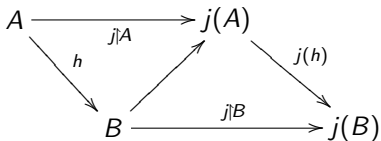


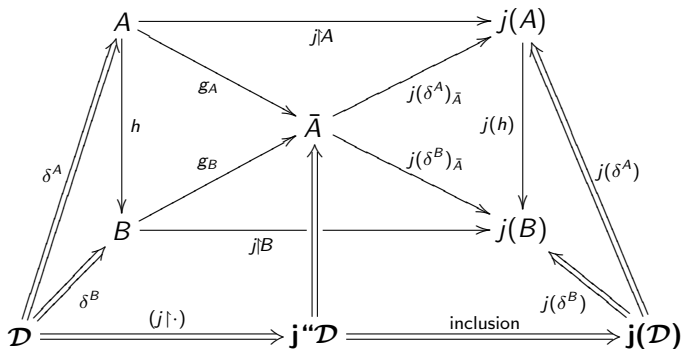


We can now extract from this the key part

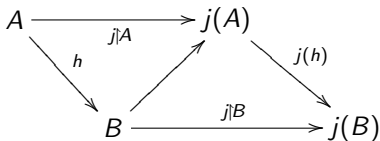


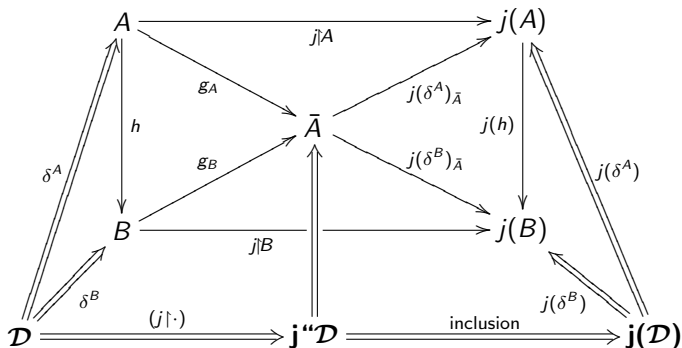
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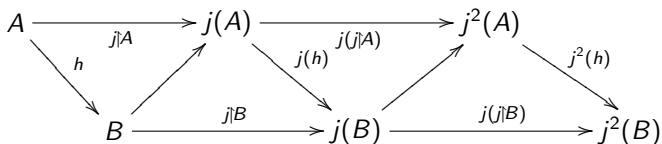


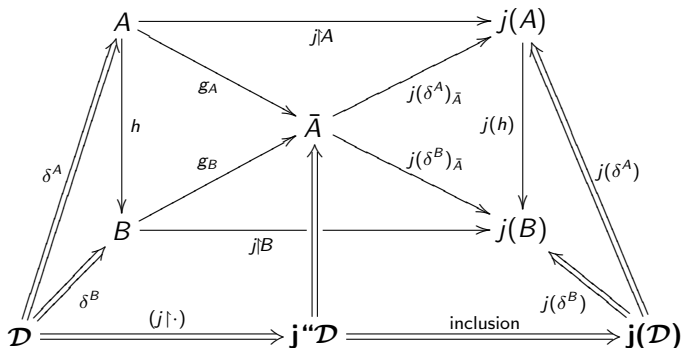
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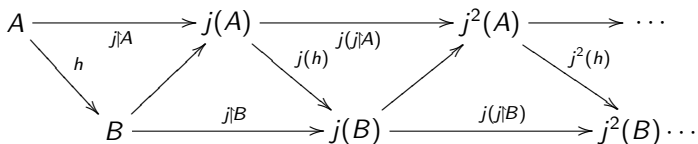


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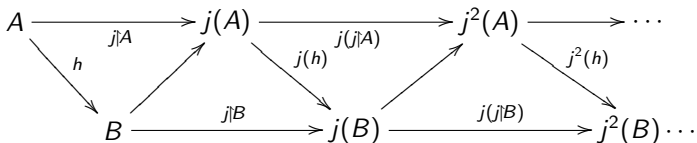




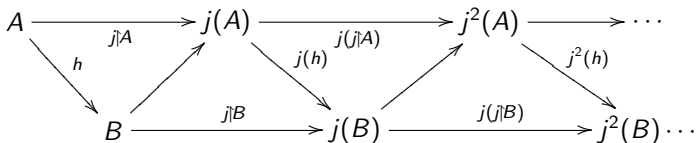
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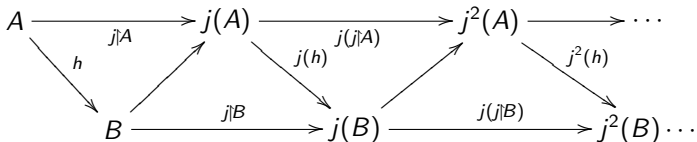
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