Combinatorial Models for Unstable and Stable Homotopy

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Homotopy Afternoon

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The homotopy category

If X and Y are topological spaces with basepoints $x_0 \in X$ and $y_0 \in Y$, a *pointed map* $f\colon X \to Y$ is a continuous function such that $f(x_0) = y_0$.

A ***homotopy*** from a pointed map $f\colon X \to Y$ to another pointed map $g\colon X \to Y$ is a continuous function $h\colon X \times [0,1] \to Y$ with $h_0 = f$ and $h_1 = g$ and $h_t(x_0) = y_0$ for all t.

We denote by $\boldsymbol{[X, Y]}$ the set of homotopy classes of pointed maps from X to Y.

The objects of the ***homotopy category*** **Ho** are the topological spaces and the set of morphisms from X to Y is $[X, Y]$. The isomorphisms in **Ho** are called *homotopy equivalences*.

Adjunction between suspension and loops

There is a natural bijection

$$[\Sigma X, Y] \cong [X, \Omega Y]$$

for all spaces X, Y, where

- $\boldsymbol{\Sigma X}$ denotes the ***reduced suspension*** of X (the space obtained by collapsing the 0 slice, the 1 slice, and the fibre over x_0 in $X \times [0, 1]$), and
- $\boldsymbol{\Omega Y}$ denotes the space of ***loops*** of Y, whose elements are the paths $\sigma\colon [0, 1] \to Y$ with $\sigma(0) = \sigma(1) = y_0$.

In categorical terms, one says that Σ is *left adjoint* to Ω, and that Ω is *right adjoint* to Σ in the homotopy category **Ho**.

Homotopy groups

Each set of the form $[X, \Omega Y]$ is a **group,** operating by composition of paths, and each set of the form $[X, \Omega\Omega Y]$ is an **abelian group.**

If S^n denotes the nth sphere, then

$$\pi_1(X) = [S^1, X] = [\Sigma S^0, X] \cong [S^0, \Omega X]$$

is the ***fundamental group*** or *Poincaré group* of X. For $n \geq 2$, the abelian groups

$$\boldsymbol{\pi_n(X) = [S^n, X]}$$

are the ***higher homotopy groups*** of X.

Whitehead's Theorem

A pointed map $f\colon X \to Y$ inducing a bijection of connected components and group isomorphisms $\pi_n(X) \cong \pi_n(Y)$ for every choice of basepoints is called a *weak equivalence*.

The **Whitehead Theorem** asserts that every weak equivalence $f\colon X \to Y$ between cell complexes is a homotopy equivalence.

A *cell complex* (or *CW-complex*) is a topological space obtained by successively attaching disks of increasing dimensions (possibly infinitely many disks) by means of suitable maps from spheres.

Polyhedra (i.e., spaces equipped with a triangulation by simplices) are cell complexes.

Categorical drawbacks of topological spaces

- The category of topological spaces **is not accessible** (there is no set of “small” topological spaces building all spaces by means of directed colimits).
- The class of cell complexes is neither closed under products nor under directed colimits.

Hence, (abstract) homotopy theorists work with **simplicial sets.**

Simplicial sets

A ***simplicial set*** is a sequence X of sets $X_0, X_1, X_2 \ldots$ together with functions

- $d_i \colon X_n \to X_{n-1}$, $i = 0, \ldots, n$ (called *faces*), and
- $s_j \colon X_n \to X_{n+1}$, $j = 0, \ldots, n$ (called *degeneracies*)

satisfying the *simplicial identities*:

- $d_i\, d_j = d_{j-1}\, d_i$ if $i < j$;
- $d_i\, s_j = s_{j-1}\, d_i$ if $i < j$;
- $d_j\, s_j = \mathrm{id} = d_{j+1}\, s_j$;
- $d_i\, s_j = s_j\, d_{i-1}$ if $i > j + 1$;
- $s_i\, s_j = s_{j+1}\, s_i$ if $i \leq j$.

The elements of X_n are called *n-simplices* of X. Those in the image of s_j for some j are called *degenerate*.

The category of simplicial sets

A *simplicial map* $f\colon X \to Y$ between simplicial sets X, Y is a sequence of functions $f_n\colon X_n \to Y_n$ commuting with faces and degeneracies.

The objects of the category **Ssets** are the simplicial sets, and the morphisms from X to Y are the simplicial maps.

The category **Ssets** is **accessible**. Thus all colimits exist in **Ssets** (taken levelwise), and all limits exist as well (levelwise). The finite simplicial sets build all simplicial sets, in the sense that every simplicial set is a directed colimit of its finite subobjects, and finite simplicial sets are $\aleph_0$*-presentable*, meaning that every map from a finite simplicial set to a directed colimit indexed by a set of cardinality $\aleph_0$ factors through some intermediate term. (**Note:** "finite" means that there is only a finite set of *nondegenerate* simplices.)

Homotopies of simplicial maps

The *interval* Δ^1, with two 0-simplices 0 and 1, and one non-degenerate 1-simplex σ with $d_0\sigma = 0$ and $d_1\sigma = 1$, plays the role of $[0, 1]$. The *simplicial n-sphere* $\Delta^n/\partial\Delta^n$ plays the role of the ordinary n-sphere.

Homotopies between simplicial maps are defined in the same way as for topological spaces.

However, if X and Y are simplicial sets, then $[X, Y]$ is **not** defined as the set of homotopy classes of simplicial maps from X to Y. This could of course be done, but would be unrelated to the homotopy category of topological spaces. We need to replace the target simplicial set Y by a **fibrant approximation.**

Standard simplices

Let Δ be the category whose objects are $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$ and whose morphisms are order-preserving functions. Then a simplicial set X is just a functor

$$X \colon \Delta^{\text{op}} \longrightarrow \textbf{Sets}.$$

Thus, **Ssets** is a category of *presheaves*, and all categories of presheaves are complete and cocomplete (i.e., all limits and colimits exist), since **Sets** is so.

As such, we may define the *standard n-simplex* Δ^n (a simplicial set) by

$$\Delta^n([m]) = \Delta([m], [n]) \qquad \text{for all } m.$$

Geometric realization

The *geometric realization* $|\Delta^n|$ of Δ^n is the *standard topological n-simplex*, and the *geometric realization* $|X|$ of a simplicial set X is the colimit of a diagram indexed by all simplicial maps $f\colon \Delta^n \to X$, taking the value $|\Delta^n|$ at each such map.

Geometric realization may be viewed as a functor from simplicial sets to topological spaces. As such, it has a right adjoint, called ***singular functor,*** sending each space X to the simplicial set $\mathbf{Sing}(\boldsymbol{X})$ whose set of n-simplices is the set of all continuous functions $|\Delta^n| \to X$.

The homotopy category of simplicial sets

Given simplicial sets X and Y, define $[X, Y]$ as the set of homotopy classes of simplicial maps from X to $\mathrm{Sing}(|Y|)$. A *weak equivalence* of simplicial sets is a map $f\colon X \to Y$ inducing bijections $[Y, B] \cong [X, B]$ for all B.

The unit $Y \to \mathrm{Sing}(|Y|)$ of the adjunction is a *fibrant approximation* of Y, while the counit $|\mathrm{Sing}(A)| \to A$ is a *cofibrant approximation* or *simplicial approximation* of a topological space A. Both are weak equivalences.

The ***homotopy category of simplicial sets*** has simplicial sets as objects and $[X, Y]$ as set of morphisms from X to Y.

Geometric realization and the singular functor define an equivalence of categories between the homotopy category of topological spaces and that of simplicial sets.

Quillen model categories

Simplicial sets form a ***Quillen model category.*** In fact, they are "models" for homotopy types of topological spaces. Moreover, they are **combinatorial models.**

- Informally, this means that a "discrete set of data" is sufficient to fully determine the homotopy type of a topological space.
- Formally, a ***combinatorial model category*** is a Quillen model category (thus, equipped with *cofibrations*, *fibrations*, and *weak equivalences* satisfying certain axioms [Quillen, 1967]) which is cocomplete, accessible, and *cofibrantly generated*.

Combinatorial model categories

Perhaps the most relevant feature of combinatorial model categories in our context is **Dugger's Lemma:** For each combinatorial model category $\mathcal{C}$ there is a regular cardinal λ such that the class of weak equivalences is closed under λ-directed colimits.

More precisely, if $X \colon I \to \mathcal{C}$ and $Y \colon I \to \mathcal{C}$ are functors where I is a partially ordered set such that every subset of less than λ elements has an upper bound, and a morphism $f \colon X \to Y$ is given such that $f_i \colon X_i \to Y_i$ is a weak equivalence for all i, then the induced map $\operatorname{colim}_I X \to \operatorname{colim}_I Y$ is also a weak equivalence.

In fact, every combinatorial model category is Quillen equivalent to a localization of a category of diagrams of simplicial sets with respect to some set of morphisms [Dugger, 2001].

We'd better have a break here...

Homology theories

An (additive) **homology theory** E_* assigns to each pair of spaces (X, A) a family of abelian groups $E_n(X, A)$ for $n \in \mathbb{Z}$, and to each map $f\colon (X, A) \to (Y, B)$ a family of group homomorphisms

$$f_n\colon E_n(X, A) \to E_n(Y, B), \quad n \in \mathbb{Z},$$

with the following properties:

- Homotopy invariance: $f \simeq g \Rightarrow f_n = g_n \; \forall n$.
- Excision: If the closure of U is contained in the interior of A, then the inclusion induces isomorphisms $E_n(X \setminus U, A \setminus U) \cong E_n(X, A)$ for all n.
- Exactness: Each pair (X, A) gives rise to a long exact sequence of abelian groups.
- Additivity: $E_n\left(\coprod_\alpha X_\alpha\right) \cong \bigoplus_\alpha E_n(X_\alpha)$ for every set of indices.

Cohomology theories

An (additive) **cohomology theory** E^* assigns to each pair of spaces (X, A) a family of abelian groups $E^n(X, A)$ for $n \in \mathbb{Z}$, and to each map $f\colon (X, A) \to (Y, B)$ a family of group homomorphisms

$$f^n\colon E^n(Y, B) \to E^n(X, A), \quad n \in \mathbb{Z},$$

with the following properties:

- Homotopy invariance: $f \simeq g \Rightarrow f^n = g^n \; \forall n$.
- Excision: If the closure of U is contained in the interior of A, then the inclusion induces isomorphisms $E^n(X \setminus U, A \setminus U) \cong E^n(X, A)$ for all n.
- Exactness: Each pair (X, A) gives rise to a long exact sequence of abelian groups.
- Additivity: $E^n\left(\coprod_\alpha X_\alpha\right) \cong \prod_\alpha E^n(X_\alpha)$ for every set of indices.

Suspension isomorphism

It follows from excision (or from Mayer–Vietoris) that:

- For every homology theory E_* there are natural isomorphisms
$$E_{n+1}(\Sigma X) \cong E_n(X) \qquad \text{for all } n \text{ and all } X.$$
- For every cohomology theory E^* there are natural isomorphisms
$$E^{n+1}(\Sigma X) \cong E^n(X) \qquad \text{for all } n \text{ and all } X.$$

Brown representability

The **Brown Representability Theorem** states that cohomology theories are representable, i.e., for each cohomology theory E^* there is a sequence of simplicial sets K_n, $n \in \mathbb{Z}$, such that

$$E^n(X) \cong [X, K_n] \qquad \text{for all } n \text{ and all } X.$$

Example: If $E^n = H^n(-; A)$ is ordinary (singular) cohomology with coefficients in an abelian group A, then $K_n = K(A, n)$ is an *Eilenberg–Mac Lane space* whose single nonzero homotopy group is A in dimension n.

The natural isomorphism $E^{n+1}(\Sigma X) \cong E^n(X)$ can thus be rewritten as $[X, \Omega K_{n+1}] \cong [X, K_n]$, and it then follows that $\Omega K_{n+1} \simeq K_n$ for all n. We say that the sequence of simplicial sets (K_n) forms a ***loop spectrum*** or Ω-*spectrum*.

Spectra

A (Bousfield–Friedlander) ***spectrum*** is a sequence K of pointed simplicial sets K_n, $n \geq 0$, equipped with *structure maps* $\Sigma K_n \to K_{n+1}$.

A spectrum K is a Σ*-spectrum* if the structure maps are weak equivalences, and a spectrum is an Ω*-spectrum* if the adjoint maps $K_n \to \Omega K_{n+1}$ are weak equivalences.

Every simplicial set X gives rise to a Σ-spectrum $X, \Sigma X, \Sigma\Sigma X, \ldots$, which is denoted by $\Sigma^\infty X$. If $X = S^0$, the resulting spectrum is called the ***sphere spectrum*** S.

The stable homotopy category

The *homotopy groups* of a spectrum are defined as

$$\pi_n(K) = \lim_{m\to\infty} \pi_{n+m}(K_m), \qquad n \in \mathbb{Z}.$$

A map $f\colon K \to L$ of spectra is a *weak equivalence* if it induces isomorphisms $\pi_n(K) \cong \pi_n(L)$ for all n.

Bousfield–Friedlander spectra from a Quillen model category, which is combinatorial. The associated homotopy category is called the ***stable homotopy category*** $\mathbf{Ho}^s$.

We may view Σ^∞ as a functor $\mathbf{Ho} \to \mathbf{Ho}^s$. As such, it has a right adjoint Ω^∞.

Additivity

A fundamental feature of spectra is that suspension sets up a natural isomorphism

$$[X, Y] \cong [\Sigma X, \Sigma Y]$$

for all spectra X and Y. In fact, $\Sigma\colon \mathbf{Ho}^s \to \mathbf{Ho}^s$ is invertible.

Therefore, the stable homotopy category $\mathbf{Ho}^s$ is ***additive:*** we may add maps between spectra, and $[X, Y]$ is an abelian group for all X and Y, since

$$[X, Y] \cong [\Sigma\Sigma X, \Sigma\Sigma Y] \cong [X, \Omega\Omega\Sigma\Sigma Y].$$

The *zero spectrum* consists of a point in all dimensions.

Triangulated categories

The stable homotopy category is a ***triangulated category.*** Namely, the suspension operator is invertible, and every map $f\colon X \to Y$ can be followed by another map $g\colon Y \to C$, where C is called the *cone* of f, and the sequence

$$X \to Y \to C \to \Sigma X \to \Sigma Y \to \Sigma C \to \Sigma\Sigma X \to \cdots$$

yields long exact sequences of abelian groups

$$[A, X] \to [A, Y] \to [A, C] \to [A, \Sigma X] \to [A, \Sigma Y] \to \cdots$$

$$\cdots \to [\Sigma Y, B] \to [\Sigma X, B] \to [C, B] \to [Y, B] \to [X, B],$$

for all spectra A and B.

The formalism of triangulated categories was introduced by Dold and Puppe in topology and by Grothendieck and Verdier in algebraic geometry in the decade of 1960.

Derived categories of rings

If R is a (commutative) ring with 1, we may consider the category $\mathrm{Ch}(R)$ of ($\mathbb{Z}$-graded) chain complexes of R-modules, and call a morphism $f\colon C \to D$ of chain complexes a *weak equivalence* if it induces homology isomorphisms $H_n(C) \cong H_n(D)$ for all n.

This yields a combinatorial Quillen model category (where the chain complex whose only nonzero term is the ring R in degree 0 is a small generator).

The corresponding homotopy category is denoted by $\boldsymbol{\mathcal{D}(R)}$ and called the ***derived category*** of the ring R.

For all R, the derived category $\mathcal{D}(R)$ is triangulated.

Ring spectra and module spectra

A spectrum K is a ***ring spectrum*** if it is equipped with maps $K \wedge K \to K$ and $S \to K$ with properties analogous to a multiplication and unit in a ring. (**Note:** In order to define the *wedge product* or *tensor product* of spectra $X \wedge Y$, the Bousfield–Friedlander model category is not suitable; one needs closed monoidal model categories, e.g. symmetric spectra, which were introduced around 2000.)

A spectrum M is a ***module spectrum*** over a ring spectrum K if it is equipped with a map $K \wedge M \to M$ with properties analogous to a module over a ring.

For each ring spectrum K, the K-module spectra form a combinatorial Quillen model category. If $K = HR$ is the spectrum representing ordinary (singular) cohomology with coefficients in a ring R, then the corresponding homotopy category is equivalent to the derived category $\mathcal{D}(R)$.

Homology and cohomology theories on spectra

Every spectrum E defines a (reduced) cohomology theory on spectra as

$$E^n(X) = [X, \Sigma^n E],$$

and also a (reduced) homology theory as

$$E_n(X) = \pi_n(X \wedge E) = [\Sigma^n S, X \wedge E].$$

- A spectrum X is ***E_*-acyclic*** if $E_n(X) = 0$ for all n; that is, if $X \wedge E \simeq 0$.
- A spectrum X is ***E^*-acyclic*** if $E^n(X) = 0$ for all n; that is, if $\mathrm{Map}(X, E) \simeq 0$, where Map denotes a function spectrum.

Example: If $K(n)$ denotes nth Morava K-theory at a prime p, then $K(n) \wedge K(m) \simeq 0$ if $n \neq m$.

Localizing and colocalizing subcategories

For a spectrum E, the classes of E_*-acyclic spectra and E^*-acyclic spectra are closed under triangles (thus under suspension and desuspension), retracts and coproducts.

A full subcategory of a triangulated category is called a *triangulated subcategory* it is closed under triangles. It is called *thick* if it is closed under retracts, and it is called a ***localizing subcategory*** if it is also closed under coproducts.

Dually, it is called ***colocalizing*** if it is triangulated and closed under retracts and products.

Hovey's problem

It is easy to prove that for every homology theory E_* there is a cohomology theory F^* such that the class of F^*-acyclics coincides with the class of E_*-acyclics. (For example, if $E_* = HA_*$ is ordinary, then $F^* = HA^*$ as well.)

Hovey's problem (1995) asks if the opposite statement is also true: Given a cohomology theory F^*, does there necessarily exist a homology theory E_* with the same acyclics?