

Reflection of some properties of uncountable structures

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- ▶ For a class \mathcal{C} of structures of a same type and a property Φ , consider the following statement:

(*) For any uncountable $A \in \mathcal{C}$, if many $B \subseteq A$ with $B \in \mathcal{C}$ and $|B| = \aleph_1$ satisfy Φ then A satisfies Φ as well.

- ▶ “many” above can mean “all”, “club many”, “stationarily many”, etc.

- ▶ (*) can be also formulated in contraposition as:

(*) For any uncountable $A \in \mathcal{C}$, if A does not satisfy Φ then there are not many $B \subseteq A$ with $B \in \mathcal{C}$ and $|B| = \aleph_1$ satisfying Φ .

Two instances of the reflection statement

Reflection of some properties (3/14)

- ▶ The following two assertions are consistent over ZFC (plus some large cardinal axiom like a strongly compact cardinal).
- ▶ Moreover, these assertions are equivalent to each other over ZFC (we do not need large cardinal here).

(†) *An uncountable boolean algebra B is openly generated if club many subalgebras of B of size \aleph_1 are openly generated.*
([S.F. and A. Rinot, 2011])

(‡) *An uncountable (undirected) graph $G = \langle G, E \rangle$ has countable coloring number if all subgraphs of G of size \aleph_1 have countable coloring number.*
([S.F., H. Sakai, L. Soukup and T. Usuba, preprint])

- ▶ There are **many** other “mathematical” reflection theorems known to be equivalent to the assertions above over ZFC.

Many equivalent reflection statements

Reflection of some properties (4/14)

- ▶ There are **many** other “mathematical” reflection theorems known to be equivalent to the assertions on the last slide over ZFC.
- ▶ The following reflection statement is also equivalent to (\dagger) and (\ddagger) :

For any locally compact Hausdorff space X if all subspace of X of cardinality $\leq \aleph_1$ are metrizable then X is also metrizable.

- The reflection 'theorems' (\dagger) and (\ddagger) are both equivalent to the following set-theoretic assertion we called **Fodor-type Reflection Principle (FRP)** in [S.F., I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba, 2010].

(FRP):

For any regular uncountable λ and any stationary $S \subseteq E_\omega^\lambda = \{\alpha < \lambda : \text{cf}(\alpha) = \omega\}$ and mapping $g : S \rightarrow [\lambda]^{\leq \aleph_0}$ there is $I \in [\lambda]^{\aleph_1}$ such that

- (1) $\text{cf}(I) = \omega_1$;
- (2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
- (3) for any regressive $f : S \cap I \rightarrow \lambda$ s.t. $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \lambda$ s.t. $f^{-1} \upharpoonright \{\xi^*\}$ is stationary in $\text{sup}(I)$.

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Under (1) and (2), (3) is equivalent to:

(3') for a/any filtration $\langle I_\alpha : \alpha < \omega_1 \rangle$ of I , the set $\{\alpha < \omega_1 : \text{sup}(I_\alpha) \in E \cap I, g(\text{sup}(I_\alpha)) \cap \text{sup}(I_\alpha) \subseteq I_\alpha\}$ is stationary in ω_1 .

- ▶ Martin's Maximum implies FRP.
- ▶ FRP holds if strongly compact cardinal is Levy collapsed to \aleph_2 .
- ▶ FRP implies the failure of the square principles.

Under Martin's Maximum we have $2^{\aleph_0} = \aleph_2$ and in the model with Levy collapse we have CH. However:

- ▶ FRP is preserved by c.c.c. generic extension. Hence
- ▷ FRP is consistent with “arbitrary” size of the continuum.

On the other hand, FRP has influence on cardinal arithmetic:

- ▶ FRP implies Shelah's Strong Hypothesis (SSH). In particular,
- ▷ FRP implies Singular Cardinal Hypothesis.
- ▶ SSH under FRP is used in the induction proof of “FRP \Rightarrow (\dagger)” for the case of successor of singular cardinal of countable cofinality.

$(\dagger) \Rightarrow \text{FRP}$: Suppose that FRP does not hold and let λ be minimal regular uncountable cardinal s.t. there are a stationary $E \subseteq E_\omega^\lambda$ and a mapping $g : E \rightarrow [\lambda]^{\leq \aleph_0}$ which are counterexamples of FRP.

► From E and g as above, we can construct an almost essentially disjoint ladder system $g^* : E^* \rightarrow [\lambda]^{\leq \aleph_0}$ s.t. $E^* \subseteq E_\omega^\lambda$ is stationary and $g(\alpha) \subseteq \lambda \setminus \text{Lim}(\lambda)$ for all $\alpha \in E^*$.

► Let $G = \langle \lambda, E \rangle$ be defined by

$$\alpha E \beta \Leftrightarrow \begin{aligned} &\alpha < \beta \text{ and } \alpha \in g^*(\beta), \text{ or} \\ &\beta < \alpha \text{ and } \beta \in g^*(\alpha). \end{aligned}$$

Then G is a counterexample of (\dagger) .

(\dagger) *An uncountable (undirected) graph $G = \langle G, E \rangle$ has countable coloring number if all subgraphs of G of size \aleph_1 have countable coloring number.*

FRP \Rightarrow (\dagger): By induction on $|G|$. We assume that all subgraphs of G of cardinality \aleph_1 have countable coloring number.

Case I.: $|G| = \aleph_1$. G is trivially of countable coloring number.

Case II.: $|G| = \lambda$ is singular. By induction hypothesis all subgraphs of G of cardinality $< \lambda$ are of countable coloring number.

► It follows from (the most general version of) Shelah Singular Compactness Theorem that G is then also of countable coloring number.

Sketch of the proof of FRP $\Leftrightarrow (\ddagger)$ (3/3)

Reflection of some properties (10/14)

Case III.: $|G| = \lambda$ is regular. Let $\langle G_\alpha : \alpha < \lambda \rangle$ be a filtration. By induction hypothesis G_α , $\alpha < \lambda$ are of countable coloring number.

► Suppose toward a contradiction that $Col(G) > \omega$. Without loss of generality, (the underlying set of) $G = \lambda$.

► By the Corollary of Erdős-Hajnal theorem,

$$E_0 = \{ \alpha < \lambda : \text{there is a } c_\alpha \in [G_\alpha]^{\aleph_0} \text{ and } \eta_\alpha \in G \setminus G_\alpha \\ \text{s.t. } \xi \in E \text{ for all } \xi \in c_\alpha \}$$
 is stationary.

► We can show $E = E_0 \cap E_\omega^\lambda \cap \{ \alpha < \lambda : G_\alpha = \alpha \}$ is stationary.

► For $\alpha \in E$ let $g(\alpha) = c_\alpha \cup \{ \eta_\alpha \}$ and apply FRP to get $I \in [\lambda]^{\aleph_1}$ s.t.

(1) $cf(I) = \omega_1$; (2) $g(\alpha) = c_\alpha \cup \{ \eta_\alpha \} \subseteq I$ for all $\alpha \in I \cap S$;

(3') for a/any filtration $\langle I_\alpha : \alpha < \omega_1 \rangle$ of I , the set $\{ \alpha < \omega_1 : \sup(I_\alpha) \in E \cap I, g(\sup(I_\alpha)) \cap \sup(I_\alpha) = c_{g(\sup(I_\alpha))} \subseteq I_\alpha \}$ is stationary in ω_1 . (see [the characterization of FRP](#))

Then $Col(G \upharpoonright I) > \omega$. But this is a contradiction to the choice of G . \square

Slight modifications of (\dagger) and (\ddagger) can be inconsistent with ZFC:

There are openly generated boolean algebras B of size $\geq \aleph_2$ which are not projective. There are club many openly generated subalgebras of B of size \aleph_1 . By definition, all these subalgebras are projective.

There is a graph G with uncountable chromatic number s.t. all subgraphs of G of cardinality $< 2^{\aleph_0}$ are of countable chromatic number. We can even replace “ $< 2^{\aleph_0}$ ” with “ $< \beth_n$ ” for any $n \in \omega$.
 ([P. Erdős and A. Hajnal, 1961])

For a cardinal $\kappa \geq (2^{\aleph_0})^+$, let $G = \langle [\kappa]^2, E \rangle$ be defined by

$$\{\alpha, \alpha'\} E \{\beta, \beta'\} \Leftrightarrow \alpha' = \beta \text{ or } \beta' = \alpha$$

for $\{\alpha, \alpha'\}, \{\beta, \beta'\} \in [\kappa]^2$ with $\alpha < \alpha'$ and $\beta < \beta'$.

Then $\text{Chr}(G) \geq \aleph_0$ but $\text{Chr}(G \upharpoonright [X]^2) = \aleph_0$ for all $X \in [\kappa]^{< 2^{\aleph_0}}$.

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for $\{\alpha, \alpha'\}, \{\beta, \beta'\} \in [\kappa]^2$ with $\alpha < \alpha'$ and $\beta < \beta'$.

Then $\text{Chr}(G) \geq \aleph_0$ but $\text{Chr}(G \upharpoonright [X]^2) = \aleph_0$ for all $X \in [\kappa]^{\leq 2^{\aleph_0}}$.

Proof. ▶ $\text{Chr}(G \upharpoonright [X]^2) = \aleph_0$ for any $X \in [\kappa]^{\leq 2^{\aleph_0}}$:

Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a family of pairwise incomparable subsets of ω with $|\mathcal{F}| = 2^{\aleph_0}$ and $f : X \rightarrow \mathcal{F}$ be an injection. For $\{\alpha, \alpha'\} \in [X]^2$ with $\alpha < \alpha'$, let $c(\{\alpha, \alpha'\}) = n_{\alpha, \alpha'}$ for some $n_{\alpha, \alpha'} \in f(\alpha) \setminus f(\alpha')$. If $c(\{\alpha, \alpha'\}) = c(\{\beta, \beta'\})$, then $\{\alpha, \alpha'\} \notin \{\beta, \beta'\}$. I.e., c is a good coloring of $G \upharpoonright [X]^2$.

▶ $\text{Chr}(G \upharpoonright [X]^2) > \aleph_0$ for any $X \in [\kappa]^{> 2^{\aleph_0}}$:

Suppose that $X \in [\kappa]^{> 2^{\aleph_0}}$ and $g : [X]^2 \rightarrow \omega$. By Erdős-Rado Theorem, there is $Y \in [X]^{\aleph_1}$ s.t. Y is homogeneous w.r.t. g . Let $\alpha, \beta, \gamma \in Y$ with $\alpha < \beta < \gamma$. Then we have $g(\{\alpha, \beta\}) = g(\{\beta, \gamma\})$ and $\{\alpha, \beta\} E \{\beta, \gamma\}$. ◻

Theorem (P. Komjath, [preprint])

- (1) (MA(Cohen)) For any graph G of cardinality \aleph_1 , we have $\aleph_0 < \text{Chr}(G) \Leftrightarrow \aleph_0 < \text{List}(G)$.
- (2) For any graph G of cardinality \aleph_1 , if $\aleph_0 < \text{Cor}(G)$ then, for $\mathbb{P} = \text{Fn}(\omega_2, 2, < \aleph_1)$, we have $\Vdash_{\mathbb{P}} \text{“} \aleph_0 < \text{List}(G) \text{”}$.

Corollary (S.F. and H. Sakai [in preparation])

- (1) “ZFC+FRP+ $\neg(\dagger)$ for list chromatic number” is consistent.
- (2) “ZFC+FRP+ (\dagger) for list chromatic number” is consistent.

Proof. (1): Let $M \models \text{ZFC} + \text{FRP}$. Let $M[G]$ be a c.c.c.-generic extension where MA(Cohen) also holds. FRP still holds in $M[G]$. Hence, by (1) of Komjath’s theorem and Erdős-Hajnal’s example we have the (strong) negation of (\dagger) for list chromatic number.

(2): Levy collapse of a strongly compact cardinal makes FRP true. So by [S.F.,H.Sakai,...] and (2) of Komjath’s theorem, (\dagger) for list chromatic number holds. \square

Problems

- ▶ Is there a reflection statement on groups which is similar to (or, e.g. equivalent to) (\dagger) ?
- ▶ Is “ZFC + FRP + (\dagger) for list chromatic number” consistent with $\neg\text{CH}$?
- ▶ Find more examples of natural reflexion statements which are independent from FRP.



$\omega + 1$

Openly generated boolean algebras

- ▶ A boolean algebra A is **projective** if $A \oplus F$ is free for a sufficiently large free boolean algebra F .
- ▶ Note that subalgebra of a boolean algebra is not necessarily free.
- ▶ A boolean algebra A is **openly generated** if $\Vdash_{\mathbb{P}}$ “ A is projective” for any σ -closed p.o. forcing the cardinality of A to be $\leq \aleph_1$.

Proposition (L. Heindorf, 198?). *A Boolean algebra A is openly generated if and only if there is a mapping $f : A \rightarrow [A]^{<\aleph_0}$ (Freese-Nation mapping) s.t., for all $a, b \in A$ with $a \leq_A b$, there is $c \in f(a) \cap f(b)$ s.t. $a \leq_A c \leq_A b$.*

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Club (stationarily) many subsets of size \aleph_1

- ▶ For a set X , let $[X]^{\aleph_1} = \{A \subseteq X : |A| = \aleph_1\}$.
- ▶ $[X]^\kappa$, $[X]^{<\kappa}$, $[X]^{\leq\kappa}$ etc. for any cardinal κ are defined similarly.
- ▶ $\mathcal{F} \subseteq [X]^{\aleph_1}$ is **club** (closed unbounded) if
 - ▶ For any upward directed $\mathcal{F}' \subseteq \mathcal{F}$ (w.r.t. \subseteq) of cardinality $\leq \aleph_1$, we have $\bigcup \mathcal{F}' \in \mathcal{F}$; and,
 - ▶ for any $A \in [X]^{\aleph_1}$ there is $B \in \mathcal{F}$ with $A \subseteq B$.
- ▶ $\mathcal{S} \subseteq [X]^{\aleph_1}$ is **stationary** if $\mathcal{S} \cap \mathcal{F} \neq \emptyset$ for all club $\mathcal{F} \subseteq [X]^{\aleph_1}$.
- ▶ “There are club many subsets of X of size \aleph_1 with ...” means “ $\{A \in [X]^{\aleph_1} : \dots\}$ contains a club subset of $[X]^{\aleph_1}$ ”.
Similarly for “stationarily many”.

Graphs with countable coloring number

- ▶ A structure $G = \langle G, E \rangle$ with $E \subseteq G^2$ is a **graph** if the binary relation E is non-reflective and symmetric. G is the set of vertices and $v E w$ (or $w E v$) if the vertices $v, w \in G$ are connected in the graph.
- ▶ G has **coloring number** $\leq \kappa$ if there is a well-ordering \sqsubset of G s.t., for all $v \in G$, we have $|\{w \in G : w \sqsubset v, w E v\}| < \kappa$
- ▶ G has **countable coloring number** if it has coloring number $\leq \aleph_0$.
- ▶ The coloring number of G (notation: $Col(G)$) is defined by
$$Col(G) = \min\{\kappa : G \text{ has coloring number } \leq \kappa\}.$$

Theorem (P. Erdős and A. Hajnal, 1961). A graph $G = \langle G, E \rangle$ has coloring number $\leq \kappa$ if and only if there is a mapping $f : G \rightarrow [G]^{<\kappa}$ s.t., for all $v, w \in G$, if $v E w$ then $v \in f(w)$ or $w \in f(v)$.

Graphs with countable coloring number (2/2)

Theorem (P. Erdős and A. Hajnal, 1961). A graph $G = \langle G, E \rangle$ has coloring number $\leq \kappa$ if and only if there is a mapping $f : G \rightarrow [G]^{<\kappa}$ s.t., for all $v, w \in G$, if $v E w$ then $v \in f(w)$ or $w \in f(v)$.

Corollary. A graph $G = \langle G, E \rangle$ has coloring number $\leq \kappa$ if and only if G has a filtration $\langle G_\alpha : \alpha < \lambda \rangle$ s.t.

- ▷ all G_α , $\alpha < \lambda$ have coloring number $\leq \kappa$; and
- ▷ for all $\alpha < \lambda$ and $v \in G \setminus G_\alpha$, $|\{w \in G_\alpha : w E v\}| < \kappa$.

Corollary. A graph $G = \langle G, E \rangle$ has coloring number $> \kappa$ if for any filtration $\langle G_\alpha : \alpha < \lambda \rangle$ of G either

- ▷ for stationarily many $\alpha < \lambda$, we have $col(G_\alpha) > \kappa$; or
- ▷ for stationarily many $\alpha < \lambda$, $|\{w \in G_\alpha : w E v_\alpha\}| \geq \kappa$ for some $v_\alpha \in G \setminus G_\alpha$.

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- ▷ for stationarily many $\alpha < \lambda$, we have $col(G_\alpha) > \kappa$; or
- ▷ for stationarily many $\alpha < \lambda$, $|\{w \in G_\alpha : w E v_\alpha\}| \geq \kappa$ for some $v_\alpha \in G \setminus G_\alpha$.

Chromatic and list chromatic number of graphs

- ▶ For a graph $G = \langle G, E \rangle$, a mapping $f : G \rightarrow \kappa$ is a **good coloring** if $f(v) \neq f(w)$ for all $v, w \in G$ with $v E w$. The **chromatic number** $Chr(G)$ of a graph G is defined by

$$Chr(G) = \min\{\kappa : \text{there is a good coloring } f : G \rightarrow \kappa\}.$$

- ▶ It is easy to see that $Chr(G) \leq Col(G)$ holds for all graph G .
- ▶ The **list chromatic number** $List(G)$ of a graph G is defined by

$$List(G) = \min\{\kappa : \text{for any } \lambda \text{ and } g : G \rightarrow [\lambda]^\kappa, \\ \text{there is a choice function } f \\ \text{which is a good coloring of } G\}$$

- ▶ We have $Chr(X) \leq List(X) \leq Col(X)$.