

Reflection of some properties of uncountable structures

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(9. September 2011 (13:57 JST) version)

**Advanced Course and Workshop
on Large-Cardinal Methods in Homotopy**
於 IMUB

September 8, 2011

This presentation is typeset by p^AT_EX with beamer class.

- ▶ For a class \mathcal{C} of structures of a same type and a property Φ , consider the following statement:

(*) For any uncountable $A \in \mathcal{C}$, if many $B \subseteq A$ with $B \in \mathcal{C}$ and $|B| = \aleph_1$ satisfy Φ then A satisfies Φ as well.

- ▶ “many” above can mean “all”, “club many”, “stationarily many”, etc.

- ▶ (*) can be also formulated in contraposition as:

(*) For any uncountable $A \in \mathcal{C}$, if A does not satisfy Φ then there are not many $B \subseteq A$ with $B \in \mathcal{C}$ and $|B| = \aleph_1$ satisfying Φ .

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Two instances of the reflection statement

Reflection of some properties (3/14)

- ▶ The following two assertions are consistent over ZFC (plus some large cardinal axiom like a strongly compact cardinal).
- ▶ Moreover, these assertions are equivalent to each other over ZFC (we do not need large cardinal here).

(†) *An uncountable boolean algebra B is openly generated if club many subalgebras of B of size \aleph_1 are openly generated.*
([S.F. and A. Rinot, 2011])

(‡) *An uncountable (undirected) graph $G = \langle G, E \rangle$ has countable coloring number if all subgraphs of G of size \aleph_1 have countable coloring number.*
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Reflection of some properties (4/14)

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For any locally compact Hausdorff space X if all subspace of X of cardinality $\leq \aleph_1$ are metrizable then X is also metrizable.

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(FRP):

For any regular uncountable λ and any stationary $S \subseteq E_\omega^\lambda = \{\alpha < \lambda : \text{cf}(\alpha) = \omega\}$ and mapping $g : S \rightarrow [\lambda]^{\leq \aleph_0}$ there is $I \in [\lambda]^{\aleph_1}$ such that

- (1) $\text{cf}(I) = \omega_1$;
- (2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
- (3) *for any regressive $f : S \cap I \rightarrow \lambda$ s.t. $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \lambda$ s.t. $f^{-1}''\{\xi^*\}$ is stationary in $\text{sup}(I)$.*

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Under (1) and (2), (3) is equivalent to:

(3') for a/any filtration $\langle I_\alpha : \alpha < \omega_1 \rangle$ of I , the set $\{\alpha < \omega_1 : \text{sup}(I_\alpha) \in E \cap I, g(\text{sup}(I_\alpha)) \cap \text{sup}(I_\alpha) \subseteq I_\alpha\}$ is stationary in ω_1 .

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Fodor-type Reflection Principle (FRP) (2/2)

Reflection of some properties (7/14)

- ▶ Martin's Maximum implies FRP.
- ▶ FRP holds if strongly compact cardinal is Levy collapsed to \aleph_2 .
- ▶ FRP implies the failure of the square principles.

Under Martin's Maximum we have $2^{\aleph_0} = \aleph_2$ and in the model with Levy collapse we have CH. However:

- ▶ FRP is preserved by c.c.c. generic extension. Hence
- ▶ FRP is consistent with “arbitrary” size of the continuum.

On the other hand, FRP has influence on cardinal arithmetic:

- ▶ FRP implies Shelah's Strong Hypothesis (SSH). In particular,
- ▶ FRP implies Singular Cardinal Hypothesis.
- ▶ SSH under FRP is used in the induction proof of “FRP \Rightarrow (\dagger)” for the case of successor of singular cardinal of countable cofinality.

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► From E and g as above, we can construct an almost essentially disjoint ladder system $g^* : E^* \rightarrow [\lambda]^{\leq \aleph_0}$ s.t. $E^* \subseteq E_\omega^\lambda$ is stationary and $g(\alpha) \subseteq \lambda \setminus \text{Lim}(\lambda)$ for all $\alpha \in E^*$.

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Sketch of the proof of FRP $\Leftrightarrow (\ddagger)$ (3/3)

Reflection of some properties (10/14)

Case III.: $|G| = \lambda$ is regular. Let $\langle G_\alpha : \alpha < \lambda \rangle$ be a filtration. By induction hypothesis G_α , $\alpha < \lambda$ are of countable coloring number.

► Suppose toward a contradiction that $Col(G) > \omega$. Without loss of generality, (the underlying set of) $G = \lambda$.

► By the Corollary of Erdős-Hajnal theorem,

$$E_0 = \{\alpha < \lambda : \text{there is a } c_\alpha \in [G_\alpha]^{\aleph_0} \text{ and } \eta_\alpha \in G \setminus G_\alpha \\ \text{s.t. } \xi \in \eta_\alpha \text{ for all } \xi \in c_\alpha\} \text{ is stationary.}$$

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There are openly generated boolean algebras B of size $\geq \aleph_2$ which are not projective. There are club many openly generated subalgebras of B of size \aleph_1 . By definition, all these subalgebras are projective.

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- (1) (MA(Cohen)) For any graph G of cardinality \aleph_1 , we have $\aleph_0 < \text{Chr}(G) \Leftrightarrow \aleph_0 < \text{List}(G)$.
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Corollary (S.F. and H. Sakai [in preparation])

- (1) “ZFC+FRP+ $\neg(\dagger)$ for list chromatic number” is consistent.
- (2) “ZFC+FRP+ (\dagger) for list chromatic number” is consistent.

Proof. (1): Let $M \models \text{ZFC} + \text{FRP}$. Let $M[G]$ be a c.c.c.-generic extension where MA(Cohen) also holds. FRP still holds in $M[G]$. Hence, by (1) of Komjath's theorem and Erdős-Hajnal's example we have the (strong) negation of (\dagger) for list chromatic number.

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- ▶ A boolean algebra A is **projective** if $A \oplus F$ is free for a sufficiently large free boolean algebra F .
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- ▶ For a set X , let $[X]^{\aleph_1} = \{A \subseteq X : |A| = \aleph_1\}$.
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Graphs with countable coloring number

- ▶ A structure $G = \langle G, E \rangle$ with $E \subseteq G^2$ is a **graph** if the binary relation E is non-reflective and symmetric. G is the set of vertices and $v E w$ (or $w E v$) if the vertices $v, w \in G$ are connected in the graph.
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$$Col(G) = \min\{\kappa : G \text{ has coloring number } \leq \kappa\}.$$

Theorem (P. Erdős and A. Hajnal, 1961). *A graph $G = \langle G, E \rangle$ has coloring number $\leq \kappa$ if and only if there is a mapping $f : G \rightarrow [G]^{<\kappa}$ s.t., for all $v, w \in G$, if $v E w$ then $v \in f(w)$ or $w \in f(v)$.*

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Chromatic and list chromatic number of graphs

► For a graph $G = (G, E)$, a mapping $f : G \rightarrow \kappa$ is a **good coloring** if $f(v) \neq f(w)$ for all $v, w \in G$ with $v E w$. The **chromatic number** $Chr(G)$ of a graph G is defined by

$$Chr(G) = \min\{\kappa : \text{there is a good coloring } f : G \rightarrow \kappa\}.$$

► It is easy to see that $Chr(G) \leq Col(G)$ holds for all graph G .

► The **list chromatic number** $List(G)$ of a graph G is defined by

$$List(G) = \min\{\kappa : \text{for any } \lambda \text{ and } g : G \rightarrow [\lambda]^\kappa, \\ \text{there is a choice function } f \\ \text{which is a good coloring of } G\}$$

► We have $Chr(X) \leq List(X) \leq Col(X)$.

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