The first Erdös cardinal $\kappa(\omega)$ and tree constructions

Definition

The first ω–Erdös cardinal $\kappa(\omega)$ is the smallest uncountable cardinal κ such that for every function f from the finite subsets of κ to 2 there exist an infinite subset $X \subseteq \kappa$ and a function $g : \omega \to 2$ such that $f(Y) = g(|Y|)$ holds for all finite subsets Y of X.

$\kappa(\omega)$ is strongly inaccessible, see
T. Jech [*Set Theory*, Monographs in Mathematics, Springer, Berlin 2002].

The trees

- If $\lambda < k(\omega)$ is a cardinal, then consider the tree $T_\lambda = {}^{\omega>}\lambda$

  $$= \{\eta : n \longrightarrow \lambda \mid n < \omega\} \text{ with bottom } \perp = \langle\emptyset\rangle.$$

- $\eta = \eta(0)^\wedge\eta(1)^\wedge \ldots {}^\wedge\eta(n-1)$ has length $\lg(\eta) = \operatorname{Dom}\eta = n$.
- Put $\eta^- = \eta(1)^\wedge \ldots {}^\wedge\eta(n-1)$.
- The ordering: If $\eta, \nu \in T$, then $\nu \subseteq \eta \iff \eta \restriction \operatorname{Dom}\nu = \nu$.
- If $\eta \in {}^{\omega\geq}\lambda$, then $[\eta] = \{\eta \restriction n \mid n < \lg(\eta)\} \subseteq T_\lambda$ denotes the support of η.
- **Old wine in new skins:** A subtree $T \subseteq T_\lambda$ is noetherian if any ascending chain of branches terminates after finitely many steps.
- Thus T is noetherian $\iff [\eta] \nsubseteq T \; \forall \eta \in {}^{\omega}\lambda$.
- T is a $(2-)$coloured tree, if $c : T \longrightarrow 2 = \{0, 1\}$ is the colouring map. Write $\mathcal{T} = (T, c)$.

Tree homomorphisms

Definition

- If T, T' are trees, then $\eta : T \longrightarrow T'$ is a homomorphism, if η preserves levels and initial segments.
- If $\mathcal{T} = (T, c), \mathcal{T}' = (T', c')$ are coloured tree, then a homomorphism $\varphi : \mathcal{T} \longrightarrow \mathcal{T}'$ is colour preserving if $c'(\eta\varphi) = c(\eta)$ for all $\eta \in T$.

The Main Theorem Part 2

Theorem

[Shelah 1982] *If* $2 \leq \kappa < \kappa(\omega) \leq \lambda$ *are cardinals,* $T_\lambda = {}^{\omega>}\lambda$ *and* $\mathcal{T}_\alpha$ $(\alpha < \lambda)$ *is a family of subtrees of* T_λ *with a* κ*-colouring* $c_\alpha : \mathcal{T}_\alpha \longrightarrow \kappa$*, then there are a generic extension of the universe and distinct* $\alpha, \beta \in \lambda$ *such that* $\mathrm{Hom}(\mathcal{T}_\alpha, \mathcal{T}_\beta) \neq \emptyset$.

Conclusion: We want to assume that $\lambda < \kappa(\omega)$

The Main Theorem Part 1

Theorem

[Shelah 1982] *If $\lambda < \kappa(\omega)$ is an infinite cardinal and $T_\lambda = {}^{\omega>}\lambda$, there is a family $\mathcal{T}_\alpha$ $(\alpha \in 2^\lambda)$ of 2-coloured subtrees of T_λ (of size λ) such that in any generic extension of the universe the following holds for $\alpha, \beta \in 2^\lambda$.*

$$\mathrm{Hom}(\mathcal{T}_\alpha, \mathcal{T}_\beta) \neq \emptyset \Longrightarrow \alpha = \beta.$$

Remark

- Such a family of coloured trees $(\mathcal{T}_\alpha, c_\alpha)$ $(\alpha < 2^\lambda)$ is called an *absolutely rigid* family of trees of size λ.
- Cosmetics: In [Shelah 1982] 2 is ω and 2^λ is λ.
- [Shelah 1982] is a long important paper: *Better quasi–orders for uncountable cardinals*, Israel J. Math. **42** (1982), 177 – 226.

Conclusion: We want a shorter proof of Part 1. [Herden 2011], [Göbel,Pokutta 2011]

Old partition principles

Essentially Silver and Rowbottom, see Drake *Set Theory*, North-Holland (1974), with some variation from Herden:

Definition

- ${}^{\omega>}\kappa^{\neq} = \{\nu : n \to \kappa \mid \nu(i) \neq \nu(i+1) \text{ for all } i+1 < n,\ n < \omega\}$
- ${}^{\omega}\kappa^{\neq} = \{\nu : \omega \to \kappa \mid \nu(i) \neq \nu(i+1) \text{ for all } i+1 < \omega\}$
- $\kappa \overset{vw}{\to} (\omega)^{<\omega}_{\lambda} \iff$ for every function $F : {}^{\omega>}\kappa^{\neq} \to \lambda$ there exists some $\eta \in {}^{\omega}\kappa^{\neq}$ with $F(\eta \restriction i) = F(\eta^{-} \restriction i)$ for all $i < \omega$.

Theorem (Silver, Fund. Math 69 (1970), 93-100.)

$\lambda < \kappa(\omega) \Longrightarrow \lambda \overset{vw}{\not\to} (\omega)^{<\omega}_{2}$.

Noetherian trees

Definition

Let $T \subseteq T_\lambda$ be a tree and $\nu \in T$. All immediate successors of ν are

$$S(\nu) = \{\rho \in T \mid \nu \subseteq \rho, \lg(\rho) = \lg(\nu) + 1\}.$$

$$T[\nu] = \{\rho \in T \mid \nu \subseteq \rho, \text{ or } \rho \subseteq \nu\} \text{ is a subtree of } T.$$

A depth function on T is a mapping $\delta : T \longrightarrow |T|^+$ such that the following holds.

- If $\nu \in T$ is maximal in T (i.e. $S(\nu) = \emptyset$), then $\delta(\nu) = 0$
- If $\nu \in T$ is not maximal, then $\delta(\nu) = \sup\{\delta(\rho) + 1 \mid \rho \in S(\nu)\}$.

Proposition

A subtree $T \subseteq T_\lambda$ is noetherian if and only if there exists a depth function $\delta : T \to |T|^+$.

Absoluteness

Lemma

Let $T_1, T_2 \subseteq T_\lambda$ with be the depth functions δ_i, if they exists. Then there is a homomorphism

$$f : T_1 \to T_2 \iff T_2 \text{ is not noetherian or}$$

$$T_1, T_2 \text{ are both noetherian with } \delta_1(\perp) \leq \delta_2(\perp).$$

Let $\mathbf{M} \subseteq \mathbf{N}$ be extensions of transitive models of ZFC.

Corollary

The class of noetherian subtrees of T_λ is absolute, which means that a tree T is noetherian in a model $\mathbf{M}$ if and only if T belongs to $\mathbf{M}$ and is noetherian in $\mathbf{N}$.

Conclusion: Now we can prove [Shelah 1982 Part 1] quickly.