Proposition: $T \subseteq T_\lambda = {}^{\omega>}\lambda$ trees

T is noetherian $\Leftrightarrow \exists$ depth function $\delta : T \to |T|^+$

Proof: "$\Leftarrow$" Suppose T is not noetherian $\Rightarrow$

$\exists\, \eta : \omega \to \lambda$ such that $[\eta] \subseteq T$

Let $\delta : T \to |T|^+$ be given $\Rightarrow$

$\delta(\eta \restriction n)$ $(n<\omega)$ is a strictly decreasing sequence

$\lightning$ ordinals

"$\Rightarrow$" Let T be noetherian.

Let $\max T$ = all maximal branches of T

$T_0 = \max T$, $T^0 = T \setminus T_0$ is a tree

Put $\delta : T_0 \to |T|^+$ $(\nu \mapsto 0)$

$T_1 = \max T^0$, $T^1 = T^0 \setminus T_1$, $\delta : T_1 \to |T|^+ (\nu \mapsto 1)$

'and so on'. At limits α, put $T^\alpha = \bigcap\limits_{\beta<\alpha} T^\beta$

$\Rightarrow \exists\, \alpha < |T|^+ \wedge T^\alpha = T^{\alpha+1}$

If $T^\alpha \neq \emptyset \Rightarrow$ construct $\eta : \omega \to \lambda \wedge [\eta] \subseteq T^\alpha$ $\lightning$

T noetherian

$\Rightarrow T^\alpha = \emptyset \Rightarrow \delta$ a total map $T \to |T|^+$

a depth function $\square$

Corollary: "Noetherian subtrees of T_λ are absolute", i.e. ...

THEOREM: If $\lambda < \kappa(\omega)$ and $T_\lambda = {}^{\omega>}\lambda$, then there exists a family $\mathbb{T}_\alpha = (T_\alpha, c_\alpha)$ $(\alpha \in 2^\lambda)$ of absolutely rigid trees $T_\alpha \subseteq T_\lambda$ with colouring $c_\alpha : T_\alpha \longrightarrow \{0,1\}$.

Proof: $\lambda < \kappa(\omega) \underset{\text{Silver}}{\Longrightarrow} \lambda \not\xrightarrow{vw} \omega_2^{<\lambda}$, i.e

$\exists\ c : {}^{\omega>}\lambda^{\neq} \longrightarrow 2$ but $\nexists\ \eta \in {}^{\omega}\lambda^{\neq}$ such that

$$c(\eta\restriction n) = c(\eta^-\restriction n) \quad \forall n<\omega \qquad \boxtimes$$

Recall $\eta \in {}^{\dots}\lambda^{\neq}$ = all neighbours $\eta(i) \neq \eta(i+1)$ are distinct

$$\eta^- = (\cancel{\eta(0)}, \eta(1), \eta(2), \dots\ )$$

$$T[\nu] = \{\rho \in T \mid \rho \subseteq \nu \text{ or } \nu \subseteq \rho\}$$

$$\alpha < \lambda \Rightarrow \langle\alpha\rangle \in T$$

Define $T(0) = \{\perp\}$, $T(n) \subseteq {}^{n}\lambda^{\neq}$, then

$$T(n+1) = \{\nu \in {}^{n+1}\lambda^{\neq} \mid \nu\restriction n, \nu^- \in T(n), c(\nu\restriction n) = c(\nu^-)\}.$$

$\Rightarrow$ $T = \bigcup_{n<\omega} T(n)$ is a tree with depth

Proposition $\Rightarrow$ T is noetherian

Define $T_\alpha = T[\langle\alpha\rangle]$ $(\alpha < \lambda)$

$\mathbb{T}_\alpha = (T_\alpha, c\restriction T_\alpha)$ $(\alpha < \lambda)$

is an absolutely rigid family of trees.

Show: rigid

Let $M \subset N$ be transitive models of ZFC

and $\mathcal{T}_\alpha$ $(\alpha < \lambda)$ a family in M

Suppose $\exists\, \beta, \gamma < \lambda$ and $f : \mathcal{T}_\beta \longrightarrow \mathcal{T}_\gamma$ homo in N.

Construct $\eta : \omega \to \lambda \wedge [\eta] \subseteq T$ (↯ noetherian)

Put $\eta(0) = \beta$

induction: Suppose $\eta \restriction n \in T_\beta$

define $\eta \restriction n+1 \in T_\beta$

with $f(\eta \restriction n) = f(\eta^- \restriction n)$

$\boxtimes \wedge f_1$ preserves colour c, and $T(n+1)$

$\Rightarrow \eta \restriction n+1 \in T_\beta \Rightarrow [\eta] \subseteq T_\beta \subseteq T$ ∎

Enlarge $\mathcal{T}_\alpha$ $(\alpha < \lambda)$:

Choose $\mathcal{F} \subseteq \mathcal{P}(\lambda)$ such that

$|\mathcal{F}| = 2^\lambda \wedge \mathcal{F}$ an antichain

(i.e. $\forall X \neq X' \in \mathcal{F} \Rightarrow X \nsubseteq X' \wedge X' \nsubseteq X$)

Define $T_X = \bigcup_{\alpha \in X} T[\langle \alpha \rangle]$, $c_X = c \restriction T_X$

$\Rightarrow$ (T_X, c_X) $(X \in \mathcal{F})$ is

absolutely rigid families of trees $\subseteq T_\lambda$.

Suppose that the branches of length $< n+1$ are constructed. Given any branch

$$\eta : n \longrightarrow \{[t_i]\dot{\cup}\{\omega, \omega+1, \omega+2, \omega+3\}$$

of length n, then ν is a successor of η of length $\lg(\nu) = n+1$ if $\nu(n)$ satisfies one of the following conditions.

- If $\eta(n-1) \notin \omega^*$ and $(n-1) \in S_\omega$, then $\nu(n) \in \eta(n-1)$.
- If $\eta(n-1) \notin \omega^*$ and $(n-1) \notin S_\omega$, then $\nu(n) = \eta(n-1)$
- If $\eta(n-1) \in \omega^*$ and $n-1 \notin S_{\eta(n-1)}$, then $\nu(n) = \eta(n-1)$.
- If $\eta(n-1) \in \omega^*$ and $n-1 \in S_{\eta(n-1)}$, then $\nu(n) \in \{\omega, \omega+1\}$.
- If $\eta(n-1) \in \{\omega, \omega+1\}$, then $\nu(n) \in \{\omega+2, \omega+3\}$.

This completes the construction of the rigid family $\mathcal{T}$ of λ..

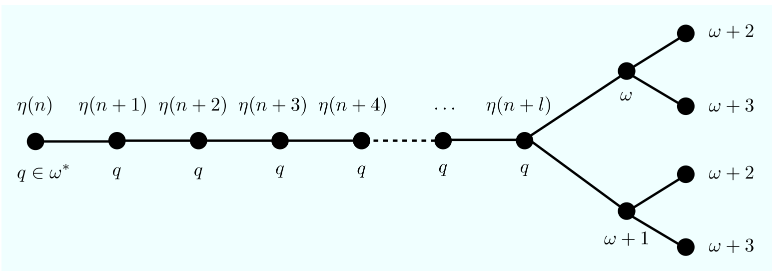

Figure 1: final-segment of a branch (here: $n+l \in S_q$)