Set Theoretical Methods Used in Abelian Groups Theory: A potential for generalization

Lecture 3

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Definition

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Theorem (Butler)

For a torsion free group of finite rank the following are equivalent:

1. *G is a pure subgroup of a completely decomposable group (of finite rank).*
2. *G is homomorphic image of a completely decomposable group of finite rank.*
3. *G can be obtained from groups of rank 1 by taking pure subgroups,finite direct sums and torsion free homomorphic image.*

A group of finite rank satisfying one of the above conditions will be called "Butler Group"

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A torsion free group G is called a B_2 group if G can be written as a continuous increasing union of pure subgroups $G = \bigcup_{\alpha<\beta} G_\alpha$ for some ordinal β such that for every $\alpha < \beta$ $G_{\alpha+1}/G_\alpha$ is of rank 1 and $G_{\alpha+1} = G_\alpha + B_\alpha$ where B_α is a Butler group of finite rank.

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Definition

Let be G be a group and $g \in G$. The G type of g is a function $\chi_G(g)$ defined on the set P of prime numbers such its values are in the set $N \cup \{\infty\}$ and such that for all $p \in P$

$$\chi_G(g)(p) = \sup\{n \in N | g \quad \text{is divisibe by } p^n\}$$

. When the group G in understood from the context we may drop the subscript G in χ_G. For types of elements in the group G we define the partial order $\leq$ by being $\leq$ for every $p \in P$.Similary we define the sup and inf of a set of types.

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Try to use filtrations . In the free case we used if H is free, $H \subseteq G$ and G/H is free then G is free and in fact we can extend every basis of H to a basis of G. Here we need : H a B_2 and G/H are B_2 groups then you can extend the sequence witnessing that is B_2 to a sequence witnessing it for G. It is problematic even in the simple case that G/H is of rank 1.

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Lemma

A pure subgroup of G, H is balanced iff for every $g \in G$ and a countable subset of H $\{h_n\}_{n<\omega}$ there is $h^* \in H$ such that for all $n \in \omega \quad \chi(h^* - h_n) \geq \chi(g - h_n)$.

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Corollary

If $\langle H_\beta | \beta < \delta \rangle$ *is an increasing sequence of balanced subgroups of G and* $\mathrm{cof}(\delta) > \omega$ *then* $\bigcup_{\beta<\delta} H_\beta$ *is a balanced subgroup of G.*

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Corollary

If $\langle H_\beta | \beta < \delta \rangle$ is an increasing sequence of balanced subgroups of G and $\mathrm{cof}(\delta) > \omega$ then $\bigcup_{\beta<\delta} H_\beta$ is a balanced subgroup of G.

Corollary

Let B be a subset of the group G then there is a balanced subgroup of G, H such that $B \subseteq H$ and $|H| \leq |B|^{\aleph_0}$.

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A subgroup of the group G,H is said to be separative in H if the set $\{\chi(g-h) | h \in H\}$ has cofinality ω, i.e. there is a countable collections of members of H ,$\{h_n\}_{n\in\omega}$ such that for every $h \in H$ there is $n \in \omega$ such that $\chi(g-h) \leq \chi(g-h_n)$.

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Examples: A balanced subgroup is separative, a countable subgroup is separative, A countable increasing union of separative subgroups is a separative subgroup,If $H \subseteq K$ are subgroups of G such that H is separative in G and K/H is separative subgroup of G/H then K is a separative subgroup of G.

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Theorem (Dugas,Hill,Ragaswamy,Fuchs)

Suppose that G is a B_1 group which is a continuous increasing union of pure subgroups: $\langle G_\alpha | \alpha < |G| \rangle$ where $G_0 = 0$,Each G_α is a B_1 separative subgroup of G and such that for every α $G_{\alpha+1}/G_\alpha$ is of rank 1 . Then G is a B_2 group.

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We shall only deal with having a filtration of any group G into separative groups which are, like in any filtration, of cardinality less than $|G|$. We are going to assume the Generalized Continuum Hypothesis (G.C.H)without it we may get into trouble even for $|G| = \aleph_2$. (Shelah and Strumgmann got a model in which CH fails and there is a B_1 group which is not B_2).

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Corollary (G.C.H)

If $|G| = \kappa$ and κ is either $\aleph_1$,a limit cardinal or a successor of a cardinal of uncountable cofinality then every group of cardinality κ admits a separative filtration.

Proof.

We shall do the case κ regular. The case κ singular can be argued similarly with minor modification. We define by induction on $\alpha < \kappa$ the filtration $\langle G_\alpha | \alpha < \kappa \rangle$. Let $\langle g_\alpha | \alpha < \kappa \rangle$ be a listing of all the members of G. At successor stage we define $G_{\alpha+1}$ to be any balanced subgroup of G containing G_α and g_α whose cardinality is at most $|G_\alpha|^{\aleph_0}$. Since by induction assumption $|G_\alpha| < \kappa$ and the assumption we made about κ $|G_{\alpha+1}| < \kappa$. $G_{\alpha+1}$ is of course separative.

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For limit α $\quad G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ is separative (If cof$(\alpha) = \omega$ the G_α is a countable increasing union of separative subgroups.If cof$(\alpha) > \omega$ then $G_\alpha = \bigcup_{\beta<\alpha} G_{\beta+1}$ hence it is balanced as an increasing union of balanced subgroup of uncountable cofinality. ☐

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Definition

λ a cardinal . $\Box_\lambda$ is the principle that claims the existence of a sequence $\langle C_\alpha | \alpha < \lambda^+, \alpha \text{ limit}\rangle$ such that

1. For $\alpha < \kappa^+$, α limit C_α is a closed unbounded subset of α of order type $\leq \kappa$
2. If β is a limit point of C_α then $C_\beta = C_\alpha \cap \beta$.

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Jensen proved that if V=L then for every cardinal λ $\Box_\lambda$ is true.

Theorem (Fuchs,M.)

If G.C.H and for every cardinal λ $\square_\lambda$ *then every group has a separative filtration . More over every* B_1 *group is a* B_2 *group. So V=L implies that every* B_1 *group is a* B_2 *group.*

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For simplifying we sketch the proof for $\kappa = \aleph_{\omega+1}$. So fix a sequence $\langle C_\alpha | \alpha < \aleph_{\omega+1} \alpha \text{ limit}\rangle$ witnessing $\Box_{\aleph_\omega}$.In this case we can assume that for $\alpha < \kappa \quad \mathrm{ot}(C_\alpha) < \aleph_\omega$.

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For simplifying we sketch the proof for $\kappa = \aleph_{\omega+1}$. So fix a sequence $\langle C_\alpha | \alpha < \aleph_{\omega+1} \alpha \text{ limit} \rangle$ witnessing $\Box_{\aleph_\omega}$.In this case we can assume that for $\alpha < \kappa \quad \text{ot}(C_\alpha) < \aleph_\omega$.
Given the group G we shall define the filtration $\langle G_\alpha | \alpha < \kappa \rangle$. For every $\alpha < \kappa$ we shall also assume G_α is an increasing union of G^n_α for $n < \omega$ where if $\text{cof}(\alpha) \neq \omega$ (including the case α successor) then each G^n_α is a balanced group of G of cardinality $\aleph_{n+1}$. Thus we are assured that G_α is separative .
(For $\text{cof}(\alpha) = \omega \quad G_\alpha$ will be a countable increasing union of separative subgroups.)

Coherence conditions

We make the following assumptions on the sequence of sequences

$$\langle\langle G^n_\alpha | n < \omega\rangle | \alpha < \kappa\rangle$$

.

1. For every $\alpha < \beta < \kappa$ there is $k < \omega$ such that $G^n_\alpha \subseteq G^n_\beta$ for $k \leq n$
2. For $\beta = \alpha + 1 \quad G^n_\alpha \subseteq G^n_\beta$ for all $n < \omega$.
3. If α is a limit ordinal and $\mathrm{ot}(C_\alpha) \geq \aleph_{n+1}$ then $G^n_{\alpha+1} = G^n_\alpha$.
4. If α is a limit point of C_β then for all $n < \omega \quad G^n_{\alpha+1} \subseteq G^n_\beta$.
5. If α is a limit point of C_β and $\mathrm{ot}(C_\alpha) \geq \aleph_{n+1}$ then $G^n_\beta = G^n_{\alpha+1} = G^n_\alpha$.

The inductive definition of

$$\langle\langle G_\alpha^n | n < \omega\rangle | \alpha < \kappa\rangle$$

involves five cases

1. $\alpha = 0$
2. α successor of a successor ordinal
3. α successor of the limit ordinal β.
4. α limit ordinal and α is not a limit point of limit points of C_α.
5. α limit ordinal and it is limit of limit points of C_α.

The use of some principle like $\Box$ is necessary because

Theorem (M., Shelah)

Assume the consistency of a supercompact cardinal, then there is a model of ZFC+GCH in which there is a group of cardinality $\aleph_{\omega+1}$ *with no separative filtration.*