

Reflecting Group Radicals and Other Cardinal Sins

Menachem Magidor

Institute of Mathematics
Hebrew University of Jerusalem

Large Cardinals Methods in Homotopy-Barcelona 2011

Joint work with J. Bagaria

outline

Group Radicals

The size of the first DG cardinal

Reflection Properties and Reflection Cardinals

Definition

Let X, A be Abelian groups. The X -radical of A is defined by

$$R_X(A) = \cap \{ \text{Ker}(f) \mid f \in \text{Hom}(A, X) \}$$

Definition

Let X, A be Abelian groups. The X -radical of A is defined by

$$R_X(A) = \cap \{ \text{Ker}(f) \mid f \in \text{Hom}(A, X) \}$$

The radical is the set of members of A such that every homomorphism of A into X sends them to 0_X .

Definition

Let X, A be Abelian groups. The X -radical of A is defined by

$$R_X(A) = \cap \{ \text{Ker}(f) \mid f \in \text{Hom}(A, X) \}$$

The radical is the set of members of A such that every homomorphism of A into X sends them to 0_X .

An Abelian group G is free* iff $R_Z(G) = \{0\}$.

If G is a subgroup of H then $R_X(G) \subseteq R_X(H)$.

If G is a subgroup of H then $R_X(G) \subseteq R_X(H)$.

Definition

Let X, H be Abelian groups. κ a cardinal. The $\kappa - X$ -radical of H is

$$R_X^\kappa(H) = \sum \{R_X(G) \mid G \subseteq H, |G| < \kappa\}$$

If G is a subgroup of H then $R_X(G) \subseteq R_X(H)$.

Definition

Let X, H be Abelian groups. κ a cardinal. The κ – X -radical of H is

$$R_X^\kappa(H) = \sum \{R_X(G) \mid G \subseteq H, |G| < \kappa\}$$

Question

Given the group X does there exists a cardinal κ such that for every group G $R_X(G) = R_X^\kappa(G)$?

If G is a subgroup of H then $R_X(G) \subseteq R_X(H)$.

Definition

Let X, H be Abelian groups. κ a cardinal. The κ – X -radical of H is

$$R_X^\kappa(H) = \sum \{R_X(G) \mid G \subseteq H, |G| < \kappa\}$$

Question

Given the group X does there exist a cardinal κ such that for every group G $R_X(G) = R_X^\kappa(G)$?

If G is a subgroup of H then $R_X(G) \subseteq R_X(H)$.

Definition

Let X, H be Abelian groups. κ a cardinal. The $\kappa - X$ -radical of H is

$$R_X^\kappa(H) = \sum \{R_X(G) \mid G \subseteq H, |G| < \kappa\}$$

Question

Given the group X does there exist a cardinal κ such that for every group G $R_X(G) = R_X^\kappa(G)$?

If such a cardinal exists then the first such cardinal is called the Dugas-Göbel cardinal of X . ("The DG cardinal of X ")

Question

If the DG of X exists, how large is it?

Question

If the DG of X exists, how large is it?

Theorem (Dugas-Göbel)

- 1. The DG cardinal of Z , if it exists is greater or equal than the first measurable cardinal.*
- 2. If κ is a strongly compact cardinal then for every group X of cardinality less than κ the DG cardinal of X exists and it is less or equal than κ .*

Definition

A cardinal κ is δ -strongly compact if every κ -complete filter on a set I can be extended to a δ complete ultrafilter.

Definition

A cardinal κ is δ -strongly compact if every κ -complete filter on a set I can be extended to a δ complete ultrafilter.

Theorem (Bagaria-M.)

A cardinal is a DG cardinal iff it is ω_1 -strong compact.

Let κ be a DG cardinal for Z . Let I be a set and \mathcal{F} be κ complete filter on I .

Consider the group $H = Z^I / \mathcal{F}$. A member of H an equivalence class modulo \mathcal{F} , $[g]_{\mathcal{F}}$ of a function $g \in Z^I$

Let κ be a DG cardinal for Z . Let I be a set and \mathcal{F} be κ complete filter on I .

Consider the group $H = Z^I / \mathcal{F}$. A member of H an equivalence class modulo \mathcal{F} , $[g]_{\mathcal{F}}$ of a function $g \in Z^I$

The map $T(g) = [g]_{\mathcal{F}}$ is a natural homomorphism of Z^I onto H . And let \hat{T} an inverse of T , namely a choice for every member of H a representative in Z^I . We assume that $\hat{T}(o_H)$ is the zero function in Z^I .

Let κ be a DG cardinal for Z . Let I be a set and \mathcal{F} be κ complete filter on I .

Consider the group $H = Z^I / \mathcal{F}$. A member of H an equivalence class modulo \mathcal{F} , $[g]_{\mathcal{F}}$ of a function $g \in Z^I$

The map $T(g) = [g]_{\mathcal{F}}$ is a natural homomorphism of Z^I onto H . And let \hat{T} an inverse of T , namely a choice for every member of H a representative in Z^I . We assume that $\hat{T}(o_H)$ is the zero function in Z^I .

Claim

$R_Z^{\kappa}(G) = \{0_H\}$ for every $G \subseteq H$ of cardinality less the κ .

claim.

We want to show that for $g \in G$ there is a homomorphism $\varphi: G \rightarrow Z$ such that $\varphi(g) \neq 0$.

claim.

We want to show that for $g \in G$ there is a homomorphism $\varphi: G \rightarrow Z$ such that $\varphi(g) \neq 0$. For $g_1, g_2 \in G$ let

$$J(g_1, g_2) = \{i \in I \mid \widehat{T}(g_1)(i) + \widehat{T}(g_2)(i) = \widehat{T}(g_1 + g_2)(i)\}$$

Note that by definition of H $J(g_1, g_2) \in \mathcal{F}$. Since \mathcal{F} is κ complete and $|G| < \kappa$ $L = \bigcap_{g_1, g_2 \in G} J(g_1, g_2)$ is in \mathcal{F} .

claim.

We want to show that for $g \in G$ there is a homomorphism $\varphi: G \rightarrow Z$ such that $\varphi(g) \neq 0$. For $g_1, g_2 \in G$ let

$$J(g_1, g_2) = \{i \in I \mid \widehat{T}(g_1)(i) + \widehat{T}(g_2)(i) = \widehat{T}(g_1 + g_2)(i)\}$$

Note that by definition of H $J(g_1, g_2) \in \mathcal{F}$. Since \mathcal{F} is κ -complete and $|G| < \kappa$, $L = \bigcap_{g_1, g_2 \in G} J(g_1, g_2)$ is in \mathcal{F} .

$g \neq 0_H$ hence $K = \{i \in I \mid g(i) = 0\} \notin \mathcal{F}$. Hence there is $i \in L - K$. Defining $\varphi(g) = g(i)$ for $g \in G$ clearly gives a homomorphism of G into Z which is not zero on g □

Hence , by κ being a DG-cardinal for Z , $R_Z(H) = \{0_H\}$. So for every $a \in H$, $a \neq 0_H$ there is a homomorphism $F : H \rightarrow Z$ such that $f(a) \neq 0_Z$.

Let $g \in Z'$ be the constant function with value 1. By the above there is a homomorphism F of H into Z such that $F(\hat{T}(g)) \neq 0_Z$. $F \circ T$ is a homomorphism of $Z' \rightarrow Z$. We invoke

Hence, by κ being a DG-cardinal for Z , $R_Z(H) = \{0_H\}$. So for every $a \in H$, $a \neq 0_H$ there is a homomorphism $F : H \rightarrow Z$ such that $f(a) \neq 0_Z$.

Let $g \in Z^I$ be the constant function with value 1. By the above there is a homomorphism F of H into Z such that $F(\hat{T}(g)) \neq 0_Z$. $F \circ T$ is a homomorphism of $Z^I \rightarrow Z$. We invoke

Theorem (Eda, Łos)

Let S be a homomorphism of Z^I into Z , then there are finitely many $U_0, U_1 \dots U_n$ σ -complete ultrafilters on I and $k_0, k_1 \dots k_n$ integers such that $S = \sum_{m=0}^n k_m \cdot F_{U_m}$ where for a σ -complete ultrafilter U on I F_U is the homomorphism of Z^I into Z defined by $F_U(g) = j$ iff $\{i \in I \mid g(i) = j\} \in U$

By the Eda-Łos theorem there are $U_0, U_1 \dots U_n$ σ -complete ultrafilters on I and $k_0, k_1 \dots k_n$ integers such that

$$F \circ T = \sum_{m=0}^n k_m \cdot F_{U_m}.$$

By the Eda-Łos theorem there are $U_0, U_1 \dots U_n$ σ -complete ultrafilters on I and $k_0, k_1 \dots k_n$ integers such that

$$F \circ T = \sum_{m=0}^n k_m \cdot F_{U_m}.$$

Since $F \circ T(g) \neq 0$ there is $m \leq n$ such that $F_{U_m}(g) \neq 0$. One can easily check that U_m must be an ultrafilter extending \mathcal{F} .

The other direction of the theorem is basically argued like in the Dugas-Göbel proof that strongly compact is a DG cardinal. For a group G one can consider the set $I = \{H \subseteq G \mid |H| < \kappa\}$. Using the fact that κ is ω_1 -strong compact one can get a ω_1 complete ultrafilter on I \mathcal{F} such that for every $g \in G$ $\{H \in I \mid g \in H\}$ is in \mathcal{F} . (It take a little argument if κ is singular.)

The other direction of the theorem is basically argued like in the Dugas-Göbel proof that strongly compact is a DG cardinal.

For a group G one can consider the set $I = \{H \subseteq G \mid |H| < \kappa\}$.

Using the fact that κ is ω_1 -strong compact one can get a ω_1 complete ultrafilter on I \mathcal{F} such that for every $g \in G$

$\{H \in I \mid g \in H\}$ is in \mathcal{F} . (It takes a little argument if κ is singular.)

If $g \notin R_Z^\kappa(G)$ then for every $H \in I$, $g \in H$ there is a homomorphism $\varphi_H : \rightarrow Z$ such that $\varphi_H(g) \neq 0$. Then we define $\varphi : G \rightarrow Z$ by

$$\varphi(g) = n \Leftrightarrow \{H \in I \mid g \in H, \varphi_H(g) = n\}$$

. φ witnesses that $g \notin R_Z(G)$.

Question

How does the first DG cardinal for Z compare with the first measurable? with the first strongly compact?

Question

How does the first DG cardinal for Z compare with the first measurable? with the first strongly compact?

If the first measurable is strongly compact then it is the first strongly compact.

Theorem

Let κ be the DG cardinal for Z then the combinatorial principle \square_λ fail for every $\lambda \geq \kappa$, the singular cardinal hypothesis holds above κ . (and if $\text{cof}(\lambda) = \omega$ then there is no good λ^+ -scale.)

Question

How does the first DG cardinal for Z compare with the first measurable? with the first strongly compact?

If the first measurable is strongly compact then it is the first strongly compact.

Theorem

Let κ be the DG cardinal for Z then the combinatorial principle \square_λ fail for every $\lambda \geq \kappa$, the singular cardinal hypothesis holds above κ . (and if $\text{cof}(\lambda) = \omega$ then there is no good λ^+ -scale.)

Theorem

If it is consistent to have a super compact cardinal then it is consistent that there are unboundedly many cardinals λ below the first supercompact for which \square_λ holds. Hence in this model the first DG cardinal is the first supercompact.

Theorem

If the existence of a supercompact cardinal is consistent then there is a model of ZFC in which there is a DG cardinal for Z which is singular. Hence it is consistent to have a ω_1 -strongly compact cardinal which is greater than the first measurable and it is not strongly compact

outline.

Start from a universe in which κ is supercompact. Force to have that the set $\{\lambda < \kappa \mid \square_\lambda\}$ is unbounded in κ while preserving the supercompactness of κ .

outline.

Start from a universe in which κ is supercompact. Force to have that the set $\{\lambda < \kappa \mid \square_\lambda\}$ is unbounded in κ while preserving the supercompactness of κ . This will guarantee that there is no DG cardinal below κ .

outline.

Start from a universe in which κ is supercompact. Force to have that the set $\{\lambda < \kappa \mid \square_\lambda\}$ is unbounded in κ while preserving the supercompactness of κ . This will guarantee that there is no DG cardinal below κ .

Let λ be a measurable cardinal less than κ . Use Radin's forcing to change the cofinality of κ to λ without collapsing cardinals.

outline.

Start from a universe in which κ is supercompact. Force to have that the set $\{\lambda < \kappa \mid \square_\lambda\}$ is unbounded in κ while preserving the supercompactness of κ . This will guarantee that there is no DG cardinal below κ .

Let λ be a measurable cardinal less than κ . Use Radin's forcing to change the cofinality of κ to λ without collapsing cardinals. You have to pick the Radin sequence carefully so that the following holds:

Fact

In the resulting model κ is still ω_1 -strongly compact, hence a DG cardinal.



reflection properties

Given a mathematical structure $\mathcal{A} = \langle M, R_0, R_1, R_2 \dots \rangle$. The structure has some property (first order, second order, whatever) a reflection property is the statement that from a certain point on in the universe of set theory the structure has a substructure of smaller cardinality having the property.

reflection properties

Given a mathematical structure $\mathcal{A} = \langle M, R_0, R_1, R_2 \dots \rangle$. The structure has some property (first order, second order, whatever) a reflection property is the statement that from a certain point on in the universe of set theory the structure has a substructure of smaller cardinality having the property. Vopenka principle is equivalent to having a reflecting cardinal to any property of structures invariant under isomorphism of the structures.

examples

Definition

An Abelian group G is free if it can be represented as direct sum of copies of Z .

$$G = \bigoplus_{i \in I} Z$$

examples

Definition

An Abelian group G is free if it can be represented as direct sum of copies of Z .

$$G = \bigoplus_{i \in I} Z$$

A subgroup of a free group is free so no problem reflecting freeness but about reflecting non freeness?

Definition

An Abelian group G is free* if it is a subgroup of a direct product of copies of Z .

$$G = \bigotimes_{i \in I} Z$$

Definition

An Abelian group G is free* if it is a subgroup of a direct product of copies of Z .

$$G = \bigotimes_{i \in I} Z$$

Trivially a subgroup of a free* group is free so we get reflection, but what about reflecting non free*?

Definition

Given a family of non empty sets \mathcal{F} . A transversal for \mathcal{F} is a one to one choice function on the sets in \mathcal{F} .

Definition

Given a family of non empty sets \mathcal{F} . A transversal for \mathcal{F} is a one to one choice function on the sets in \mathcal{F} .

If we have a transversal for a family \mathcal{F} then trivially we have a transversal for every subfamily of \mathcal{F} so get reflection for the existence of transversal. What about reflecting the non existence of transversals?

Definition

Given a family of non empty sets \mathcal{F} . A transversal for \mathcal{F} is a one to one choice function on the sets in \mathcal{F} .

If we have a transversal for a family \mathcal{F} then trivially we have a transversal for every subfamily of \mathcal{F} so get reflection for the existence of transversal. What about reflecting the non existence of transversals? (in order to avoid trivialities we assume some bound of the size of sets in \mathcal{F} , say assume the sets in \mathcal{F} are all countable.

Topological example

A subspace of a metric space is metric, so we have reflection for the space being metric. Do we have reflection for the space being non metric?

Topological example

A subspace of a metric space is metric, so we have reflection for the space being metric. Do we have reflection for the space being non metric? In order to avoid trivial answer we assume that the space is first countable

Consider the property of a graph of having uncountable chromatic number. Can we have reflection for this property?

Reflection cardinal

Definition

Let \mathcal{C} be a class of structures in the signature closed under isomorphism. Let \mathcal{P} be a subclass of \mathcal{C} also closed under isomorphism. A cardinal κ is a reflection cardinal for the property \mathcal{P} of the class \mathcal{C} if every structure in \mathcal{P} has a substructure of cardinality $< \kappa$ which is in \mathcal{P} .

All the previous examples were second order properties of the relevant structure.

All the previous examples were second order properties of the relevant structure.

Theorem

There is a reflection cardinal for every second order property iff there is a supercompact cardinal

But for each of the properties listed above an ω_1 - strongly compact cardinal is sufficient for reflection.

But for each of the properties listed above an ω_1 - strongly compact cardinal is sufficient for reflection.

Question

For which of the properties above the implication "If κ is a ω_1 -strong compact then it is reflecting for the given property" can be reversed?

The results about the group radicals show :

Theorem

κ is a reflection cardinal for the property of a group not being free iff κ is ω_1 -strong compact.*

The results about the group radicals show :

Theorem

κ is a reflection cardinal for the property of a group not being free iff κ is ω_1 -strong compact.*

This is because a group G is free* iff $R_Z(G) = 0$.

An example of another equivalence to ω_1 -strong compactness.

Definition

A topological space X is λ -compact or λ -Lindelof iff every cover by X by open set has a subcover of size $< \kappa$, so compact in the usual sense is being \aleph_0 -Lindelof and being Lindelof is being ω_1 -Lindelof.

An example of another equivalence to ω_1 -strong compactness.

Definition

A topological space X is λ -compact or λ -Lindelof iff every cover by X by open set has a subcover of size $< \kappa$, so compact in the usual sense is being \aleph_0 -Lindelof and being Lindelof is being ω_1 -Lindelof.

Theorem

κ is ω_1 -strong compact iff the product of any number of Lindelof spaces is κ compact.

An example of another equivalence to ω_1 -strong compactness.

Definition

A topological space X is λ -compact or λ -Lindelof iff every cover by X by open set has a subcover of size $< \kappa$, so compact in the usual sense is being \aleph_0 -Lindelof and being Lindelof is being ω_1 -Lindelof.

Theorem

κ is ω_1 -strong compact iff the product of any number of Lindelof spaces is κ compact.

The existence of supercompact cardinal is equivalent to having a reflecting cardinal for every second order property that can be expressed by a formula of the form : $\forall X \exists y \forall z \Phi(X, y, z)$ where Φ is quantifier free. (Upper case variables are second order and lower case are first order .)

The existence of supercompact cardinal is equivalent to having a reflecting cardinal for every second order property that can be expressed by a formula of the form : $\forall X \exists y \forall z \Phi(X, y, z)$ where Φ is quantifier free. (Upper case variables are second order and lower case are first order .)

Theorem

The existence of a ω_1 -strongly compact cardinal is equivalent to having a reflecting cardinal for every property of the form $\forall X \exists y \Phi(X, y)$ where Φ is of the form $\bigwedge_{i \in \omega} \Psi_i(X, y)$ where Ψ_i are quantifier free finitary formula.

The existence of supercompact cardinal is equivalent to having a reflecting cardinal for every second order property that can be expressed by a formula of the form : $\forall X \exists y \forall z \Phi(X, y, z)$ where Φ is quantifier free. (Upper case variables are second order and lower case are first order .)

Theorem

The existence of a ω_1 -strongly compact cardinal is equivalent to having a reflecting cardinal for every property of the form $\forall X \exists y \Phi(X, y)$ where Φ is of the form $\bigwedge_{i \in \omega} \Psi_i(X, y)$ where Ψ_i are quantifier free finitary formula.

All the examples above can be expressed in this form.

It is clear that if κ is λ -strongly compact then it is greater or equal than the DG cardinal for every group of cardinality $< \lambda$

It is clear that if κ is λ -strongly compact then it is greater or equal than the DG cardinal for every group of cardinality $< \lambda$

Question

Given a cardinal λ . Is there a group of cardinality $< \lambda$ such that being λ -strongly compact is equivalent to being greater or equal than the DG cardinal for G ?

It is clear that if κ is λ -strongly compact then it is greater or equal than the DG cardinal for every group of cardinality $< \lambda$

Question

Given a cardinal λ . Is there a group of cardinality $< \lambda$ such that being λ -strongly compact is equivalent to being greater or equal than the DG cardinal for G ?

Question

Find more reflection properties which are equivalent to the existence of a ω_1 -strongly compact cardinal.

It is clear that if κ is λ -strongly compact then it is greater or equal than the DG cardinal for every group of cardinality $< \lambda$

Question

Given a cardinal λ . Is there a group of cardinality $< \lambda$ such that being λ -strongly compact is equivalent to being greater or equal than the DG cardinal for G ?

Question

Find more reflection properties which are equivalent to the existence of a ω_1 -strongly compact cardinal.

In particular:

Conjecture

There is a reflecting cardinal for the property a first countable topological space being non-metrizable