Problem Session

Monday, September 5

Most of the questions below refer to the stable homotopy category (i.e., the homotopy category of spectra). However, they can also be considered in other triangulated homotopy categories of combinatorial model categories; in particular in derived categories of commutative rings with 1.

Recall that, if A and B are abelian groups, and HA, HB denote the corresponding Eilenberg–Mac Lane spectra, then

$$[HA, \Sigma^n HB] = \begin{cases} \mathrm{Hom}(A,B) & \text{if } n = 0; \\ \mathrm{Ext}(A,B) & \text{if } n = 1; \\ 0 & \text{if } n < 0. \end{cases}$$

In a triangulated category with products and coproducts, a full subcategory is

- *semilocalizing* if it is closed under cofibres, extensions, and coproducts;
- *semicolocalizing* it it is closed under fibres, extensions, and products;
- *localizing* if it is triangulated and closed under coproducts;
- *colocalizing* it it is triangulated and closed under products.

For each object E of a triangulated category with products and coproducts,

- the class ${}^{\lrcorner}E = \{X \ : \ [X, \Sigma^k E] = 0 \text{ for } k \leq 0\}$ is semilocalizing;
- the class $E^{\llcorner} = \{Y \ : \ [\Sigma^k E, Y] = 0 \text{ for } k \geq 0\}$ is semicolocalizing;
- the class ${}^{\perp}E = \{X \ : \ [X, \Sigma^k E] = 0 \text{ for all } k \in \mathbb{Z}\}$ is localizing;
- the class $E^{\perp} = \{Y \ : \ [\Sigma^k E, Y] = 0 \text{ for all } k \in \mathbb{Z}\}$ is colocalizing.

A full subcategory of abelian groups is a

- *torsion class* if it is closed under quotients, extensions, and coproducts;
- *torsion-free class* if it is closed under subgroups, extensions, and products.

A full subcategory $\mathcal{S}$ of a category $\mathcal{C}$ is called *reflective* if the inclusion of $\mathcal{S}$ into $\mathcal{C}$ has a left adjoint (called a *localization*) and it is called *coreflective* if the inclusion of $\mathcal{S}$ into $\mathcal{C}$ has a right adjoint (called a *colocalization*). If $\mathcal{C}$ is triangulated and a localization L preserves triangles, then the full image of L is a colocalizing subcategory and the kernel of L is a localizing subcategory. Dual statements hold for a colocalization.

In the category of abelian groups, all torsion classes are coreflective and all torsion-free classes are reflective.

Vopěnka's Principle (VP) has several equivalent statements:

- There is no rigid proper class of graphs.
- No accessible category has a rigid proper class of objects.
- **Ord** cannot be fully embedded into any accessible category.
- Every full subcategory closed under colimits in a cocomplete accessible category is coreflective.
- Every full subcategory closed under limits in a cocomplete accessible category is the orthogonal complement of some set of morphisms.

We call *Weak Vopěnka's Principle* (WVP) the following equivalent statements:

- **Ord**$^{\mathrm{op}}$ cannot be fully embedded into any accessible category.
- Every full subcategory closed under limits in a cocomplete accessible category is reflective.

In what follows, CGR refers to Casacuberta–Gutiérrez–Rosický, and BCMR refers to Bagaria–Casacuberta–Mathias–Rosický. We also thank Boris Chorny, Adam Przeździecki, José Luis Rodríguez, Greg Stevenson and Luke Wolcott for their contributions and comments to this list of problems.

Questions

(1) *Are all semicolocalizing subcategories of spectra reflective?*

CGR: An affirmative answer is implied by VP.

Przeździecki: An affirmative answer implies WVP. Indeed, if we assume the negation of WVP, then there is a full subcategory of abelian groups which is closed under limits but not reflective.

(2) ***Is WVP equivalent to the claim that every semicolocalizing subcategory of spectra is reflective?***

(3) *Is WVP equivalent to VP? Is WVP equivalent to the existence of a proper class of measurable cardinals?*

(4) *Consider the statement that every full subcategory closed under products and retracts in a cocomplete accessible category is weakly reflective. Is this statement equivalent to WVP or to VP?*

BCMR: Weak reflectivity of the closure under products and retracts of the class of groups $\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}$ for all cardinals κ lies between the existence of a supercompact cardinal and the existence of a measurable cardinal.

(5) ***Are all colocalizing subcategories of spectra reflective?***

CGR: An affirmative answer is implied by VP.

Is there a counterexample if $V = L$?

(6) ***Is there a set or a proper class of (co)localizing subcategories of spectra?***

Ohkawa: There is only a set of homological Bousfield classes.

Stevenson: If there is only a set of (co)localizing subcategories, then each of them is singly generated.

Stanley: There is a proper class of semicolocalizing subcategories of spectra. In fact, there are arbitrarily large rigid sets of abelian groups; hence there is a proper class of nullity classes of spectra. (A *nullity class* is one of the form $E^{\llcorner}$ for some spectrum E. Note that a cohomological Bousfield class is one of the form ${}^{\perp}E$ for some spectrum E.)

(7) *Is there a one-to-one correspondence between localizing and colocalizing subcategories of spectra?*

CGR: The answer is affirmative under VP.

Neeman: The answer is affirmative in derived categories of Noetherian rings.

(8) ***Are all cohomological Bousfield classes of spectra also homological Bousfield classes?*** (Hovey's Problem)

BCMR: Homological Bousfield classes are Δ_1 and cohomological Bousfield classes are Δ_2 (with parameters).

BCMR: Σ_1 semilocalizing subcategories of spectra are coreflective. If there is a proper class of supercompact cardinals, then Σ_2 semilocalizing subcategories are coreflective.

(9) *Can the statement that all cohomological Bousfield classes are coreflective be proved in ZFC?* (That is, do cohomological localizations exist in ZFC?)

BCMR: The truth of this statement follows from the existence of a proper class of supercompact cardinals.

(10) *Is every localizing subcategory of spectra a homological Bousfield class?*

Stevenson: There exist tensored triangulated categories where the answer is negative. The answer is affirmative in derived categories of Noetherian commutative rings.

(11) *Is there a set or a proper class of cohomological Bousfield classes of spectra?*

(12) *Is there a rigid proper class of spectra?* (In the enriched sense, i.e., such that $[X_i, \Sigma^n X_j] = 0$ for all $i \neq j$ and $n \in \mathbb{Z}$.)

(13) ***Is every semicolocalizing subcategory of spectra singly generated?*** (A semicolocalizing subcategory $\mathcal{L}$ is *generated* by an object $X \in \mathcal{L}$ if $\mathcal{L}$ is the smallest semicolocalizing subcategory containing X.)

CGR: If VP holds, then every semilocalizing subcategory of spectra is singly generated.

Every homological Bousfield class is singly generated; indeed, $\langle X \rangle$ is generated by aX, where a denotes Bousfield's complement operator.

Göbel–Shelah: If VP holds, then all torsion classes of abelian groups are singly generated.

Dugas–Göbel: The class of $\aleph_1$-free abelian groups is a torsion-free class which is not singly generated.

(14) *What large-cardinal strength is needed to prove that every left-proper combinatorial model category admits Bousfield localizations at arbitrary classes of morphisms?*

Rosický–Tholen: VP implies this fact.

Przeździecki: The negation of WVP yields a counterexample.

BCMR: The existence of a proper class of supercompact cardinals suffices under mild additional assumptions.

(15) ***Does one need large cardinals to prove that every directed colimit of accessible categories and accessible embeddings is accessible?***

Rosický: The existence of arbitrarily large compact cardinals implies this fact.

(16) *Does one need large cardinals to prove that the full image of every accessible functor is preaccessible?* (A *preaccessible* category is the same as an accessible category, but dropping the assumption that directed colimits exist.)

Rosický: The existence of arbitrarily large compact cardinals implies this fact.

(17) *Let $\mathcal{K}$ be a stable simplicial model category and $\mathcal{C}$ be a full subcategory of Ho$\mathcal{K}$. Is it true that $\mathcal{C}$ is semilocalizing if and only if it is closed under homotopy colimits? And is it true that $\mathcal{C}$ is semicolocalizing if and only if it is closed under homotopy limits?*

CGR: Every semilocalizing subcategory is closed under homotopy colimits of objectwise cofibrant diagrams. (The homotopy colimit functor is left derived from the colimit functor.)

References

J. Bagaria, C. Casacuberta, A. R. D. Mathias and J. Rosický, Definable orthogonality classes in accessible categories are small, arXiv:1101.2792.

C. Casacuberta, J. J. Gutiérrez and J. Rosický, Are all localizing subcategories of stable homotopy categories coreflective?, arXiv:1106.2218.

M. Dugas and R. Göbel, On radicals and products, *Pacific J. Math.* **118** (1985), 79–104.

R. Göbel and S. Shelah, Semirigid classes of cotorsion-free abelian groups, *J. Algebra* **93** (1985), 136–150.

M. Hovey, Cohomological Bousfield classes, *J. Pure Appl. Algebra* **103** (1995), 45–59.

A. Neeman, Colocalizing subcategories of $\mathcal{D}(R)$, *J. reine angew. Math.*, to appear.

A. Przeździecki, An "almost" full embedding of the category of graphs into the category of abelian groups, arXiv:1104.5689.

J. Rosický and W. Tholen, Left-determined model categories and universal homotopy theories, *Trans. Amer. Math. Soc.* **355** (2003), 3611–3623.

D. Stanley, Invariants of t-structures and classification of nullity classes, *Adv. Math.* **224** (2010), 2662–2689.