

# How comprehensive is the category of abelian groups?

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## When a category is comprehensive?

When the category of graphs fully embeds into it.

### Why graphs?

Every accessible category admits a full embedding into graphs (Adámek-Rosický, 1994)

Assuming that the measurable cardinals are bounded above every concrete category fully embeds into graphs (Hedrlín-Kučera, 1969 unpublished)

Some categories admit a full embedding of *Graphs*:

- ▶ Category of semigroups (Hedrlín-Lambek, 1969)
- ▶ Category of integral domains (Fried-Sichler, 1977)

Some more categories admit an “almost” full embedding of *Graphs*:

- ▶ Category of metric spaces (up to constant maps – Trnková, 1972)
- ▶ Category of paracompact spaces (up to constant maps – Koubek, 1974)
- ▶ Category of groups (up to trivial homomorphisms and conjugation in the targets)
- ▶ The unpointed homotopy category (up to null-homotopic maps)
- ▶ Category of abelian groups (up to ???)

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and conjugation in the targets )
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- ▶ Category of abelian groups (up to ???)

$$G : \mathcal{G}raphs \rightarrow \mathcal{A}b$$

- ▶ full embedding:

$$\text{Hom}(X, Y) \xrightarrow{\cong} \text{Hom}(GX, GY)$$

- ▶ “almost” full for abelian groups:

$$\mathbb{Z}[\text{Hom}_{\mathcal{G}raphs}(X, Y)] \xrightarrow{\cong} \text{Hom}_{\mathcal{A}b}(GX, GY)$$

where  $\mathbb{Z}[S]$  is the free group with basis  $S$ .

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Natural completion of abelian groups  $\eta_A : A \rightarrow \widehat{A}$ .

- ▶  $\widehat{A} \cong \lim_{n \in \mathbb{N}} A/nA$
- ▶  $\widehat{A} \cong \prod_p A_p^\wedge$
- ▶  $\ker \eta_A$  is the divisible subgroup of  $A$ .

The completion is an idempotent functor (localization, reflector), that is:

- ▶  $\widehat{\eta}_A : \widehat{A} \xrightarrow{\cong} \widehat{A}$
- ▶ For all groups  $A, B$  the map

$$\mathrm{Hom}(\widehat{A}, \widehat{B}) \xrightarrow{\eta_A} \mathrm{Hom}(A, \widehat{B})$$

is an isomorphism.

If  $A \subseteq C \subseteq \widehat{A}$  and  $C$  is pure in  $\widehat{A}$  then  $\widehat{C} = \widehat{A}$ . In particular the inclusion  $A \subseteq C$  induces an isomorphism

$$\mathrm{Hom}(A, \widehat{A}) \cong \mathrm{Hom}(C, \widehat{A})$$

## Theorem (Corner, 1963)

Let  $A$  be a ring of cardinality at most continuum, whose additive group is free. Then there exists a group  $\mathbb{A}$  such that

- (a)  $A \subseteq \mathbb{A} \subseteq \widehat{A}$  as left  $A$ -modules.
- (b)  $A \cong \text{Hom}(\mathbb{A}, \mathbb{A})$
- (c)  $|A| = |\mathbb{A}|$

The construction:

Choose elements  $\alpha_a, \beta_a$  ( $a \in A$ ) of  $\widehat{\mathbb{Z}}$  that are algebraically independent over  $\mathbb{Z}$ . Define elements  $e_a$  ( $a \in A$ ) of  $\widehat{A}$  as

$$e_a = \alpha_a \cdot 1 + \beta_a \cdot a$$

and take  $\mathbb{A}$  to be the pure subgroup of  $\widehat{A}$  generated by  $A$  and  $Ae_a$  ( $a \in A$ ).

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## Construction of the functor $G$

Let  $\Gamma$  be a full subcategory of *Graphs* whose objects are representatives of countable graphs.

Let  $A = \mathbb{Z}[\Gamma]$  be the ring whose additive group is free with the basis consisting of the identity 1 and the maps  $\varphi : X \rightarrow Y$  in  $\Gamma$ .

If  $\psi : M \rightarrow N$  is another map in  $\Gamma$  then

- ▶ the product  $\varphi\psi$  in  $A$  is the composition  $\varphi\psi$  if  $N = X$
- ▶  $\varphi\psi = 0$  if  $N \neq X$

$$A \cong \text{Hom}(\mathbb{A}, \mathbb{A})$$

The identity  $\text{id}_X : X \rightarrow X$  in  $\Gamma$  is an idempotent of  $A$  hence we have  $\mathbb{A} \cong \text{id}_X \cdot \mathbb{A} \oplus (1 - \text{id}_X) \cdot \mathbb{A}$ .

Define:

- ▶ On  $\Gamma$ :  $GX = \text{id}_X \cdot \mathbb{A}$ ,  $G\varphi =$  left multiplication by  $\varphi$ .
- ▶ Extend  $G$  to all countable graphs.
- ▶ For arbitrary graph  $X$ , let  $[X]$  be the poset of countable subgraphs  $C \subseteq X$ , define:

$$GX = \text{colim}_{C \in [X]} GC$$

# Proof of $\mathbb{Z}[\text{Hom}_{\text{Graphs}}(X, Y)] \xrightarrow{\cong} \text{Hom}_{\text{Ab}}(GX, GY)$

Thanks to the  $A$ -module inclusions  $A \subseteq \mathbb{A} \subseteq \widehat{A}$ , the proof is almost tautological.

**P1** Let  $\Gamma_X$  denote the set of maps  $C \rightarrow X$  in  $\Gamma$ . Left multiplication by  $\text{id}_X$  is the identity on  $\Gamma_X$  and zero on  $1 - \text{id}_X$  and  $\Gamma \setminus \Gamma_X$ . We obtain:

$$\langle \Gamma_X \rangle \subseteq GX \subseteq \langle \widehat{\Gamma_X} \rangle$$

**P2** Every  $u \in GX$  may be uniquely written as the finite sum

$$u = \sum z_i \sigma_i$$

where  $\sigma_i \in \Gamma_X$  and  $z_i \in \widehat{\mathbb{Z}}$ .

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**P3** If  $\varphi : X \rightarrow Y$  is one-to-one then so is  $G\varphi : GX \rightarrow GY$ .

$\varphi : \widehat{\langle \Gamma_X \rangle} \rightarrow \widehat{\langle \Gamma_Y \rangle}$  is a monomorphism  
 directed colimits preserve monomorphisms

**P4** Every homomorphism  $h : GX \rightarrow GY$  with  $X$  and  $Y$  in  $\Gamma$  is uniquely represented as left multiplication by an  $a = \sum k_i \sigma_i$  in  $A$  where  $k_i$  are nonzero integers and  $\sigma_i : X \rightarrow Y$  are distinct maps in  $\Gamma$ . That is in  $\Gamma$  we have:

$$\mathbb{Z}[\text{Hom}(X, Y)] \cong \text{Hom}(GX, GY)$$

The composition  $\mathbb{A} \xrightarrow{\text{id}_X \cdot -} GX \xrightarrow{h} GY \subseteq \mathbb{A}$  is represented this way.

**P5** For  $X, Y$  in  $\Gamma$  and any  $h : GX \rightarrow GY$  if  $h(\text{id}_X) = 0$  then  $h = 0$ .  
 $h(x) = ax$ ,  $a$  as above, and  $h(\text{id}_X) = a$ .

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**P6** Let  $W$  and  $\varphi : X \rightarrow Y$  be in  $\Gamma$ .

If

$$\begin{array}{ccc}
 \{\text{id}_W\} & \xrightarrow{\tilde{h}} & GX \\
 \text{in} & \nearrow & \downarrow G\varphi \\
 GW & \xrightarrow{h} & GY
 \end{array}$$

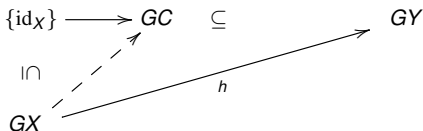
Then the dashed arrow exist.

$$h(\text{id}_W) = G\varphi u, \quad u = \sum z_j \tau_j$$

$$\sum k_i \sigma_i = \sum z_j \varphi \tau_j$$

**P7** If  $X$  is countable then for any homomorphism  $h : GX \rightarrow GY$  there exists a countable subgraph  $C \subseteq Y$  such that  $h$  factors through  $GC \subseteq GY$ .

$GX = \text{colim}_{C \in [X]} GC$  is a directed colimit  $\implies \text{id}_X \in GC$  for some countable  $C$ .

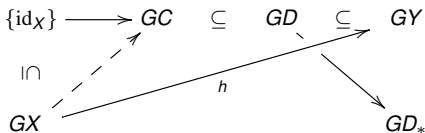


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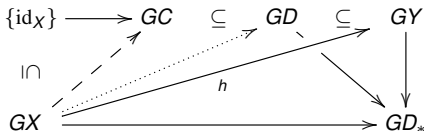


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**P8** If  $X$  is countable then

$$\mathbb{Z}[\text{Hom}(X, Y)] \cong \text{Hom}(GX, GY)$$

is an isomorphism.

P4  $\implies$  isomorphism when  $Y$  is also countable

P7 and directedness of the colimit  $\implies$  the isomorphism

$$\mathbb{Z}[\text{Hom}(X, Y)] \cong \mathbb{Z}[\text{Hom}(X, \text{colim}_{C \in [Y]} C)] \cong \text{colim}_{C \in [Y]} \mathbb{Z}[\text{Hom}(X, C)] \xrightarrow{\cong} \mathbb{Z}[\text{Hom}(X, Y)]$$

$\uparrow$   
P4

$$\rightarrow \text{colim}_{C \in [Y]} \text{Hom}(GX, GC) \xrightarrow{\cong} \text{Hom}(GX, \text{colim}_{C \in [Y]} GC) \cong \text{Hom}(GX, GY).$$

$\uparrow$   
P7

**P9** Let  $\{S_i\}_{i \in I}$  be a diagram of sets.

We obtain  $\lambda : \mathbb{Z}[\lim S_i] \rightarrow \lim \mathbb{Z}[S_i]$

- ▶ If  $I$  is codirected then  $\lambda$  is one-to-one.
- ▶ If  $I$  is countably codirected then  $\lambda$  is an isomorphism.

## Theorem

For any graphs  $X$  and  $Y$

$$\mathbb{Z}[\text{Hom}(X, Y)] \xrightarrow{\cong} \text{Hom}(GX, GY)$$

## Proof.

$$\mathbb{Z}[\text{Hom}(X, Y)] \cong \mathbb{Z}[\text{Hom}(\text{colim}_{C \in [X]} C, Y)] \cong \mathbb{Z}[\lim_{C \in [X]} \text{Hom}(C, Y)] \xrightarrow{\lambda}$$

$$\rightarrow \lim_{C \in [X]} \mathbb{Z}[\text{Hom}(C, Y)] \xrightarrow{\lim \gamma} \lim_{C \in [X]} \text{Hom}(GC, GY) \cong$$

$$\cong \text{Hom}(\text{colim}_{C \in [X]} GC, GY) \cong \text{Hom}(GX, GY).$$

P9  $\implies$   $\lambda$  is an isomorphism

P8  $\implies$   $\lim \gamma$  is an isomorphism



## Finite approximation

Let  $\Gamma_{fin} \subseteq \Gamma$  category of finite graphs

$$\mathbf{A}_{fin} = \mathbb{Z}[\Gamma_{fin}] = \text{Hom}(\mathbb{A}_{fin}, \mathbb{A}_{fin}), \quad \mathbf{G}_{fin}X = \text{colim}_{C \in [X]_{fin}} \mathbf{G}_{fin}C$$

$$\gamma : \mathbb{Z}[\text{Hom}_{\text{Graphs}}(X, Y)] \rightarrow \text{Hom}(\mathbf{G}_{fin}X, \mathbf{G}_{fin}Y).$$

1. If  $X$  is infinite then  $|X| = |\mathbf{G}_{fin}X|$ .
2. The inclusion of  $\mathbf{A}_{fin}$ -modules  $\mathbb{A}_{fin} \subseteq \mathbb{A}$  yields a natural transformation  $h : \mathbf{G}_{fin} \rightarrow G$  consisting of inclusions.
3.  $\gamma$  is one-to-one.
4. If  $X$  is finite then  $\gamma$  is an isomorphism.
5. If  $\text{Hom}(X, Y)$  is finite then  $\gamma$  is an isomorphism.

## Rigid systems of groups

### Theorem (Shelah, 1974)

*For any infinite cardinal  $\kappa$  there exists a system of groups  $\{M_i\}_{i < 2^\kappa}$  such that  $|M_i| = \kappa$  and if  $h : M_i \rightarrow M_j$  is a nonzero homomorphism then  $i = j$  and  $h$  is a multiplication by an integer.*

### Alternative proof.

*Vopěnka and Hedrlín (1965) proved that for any infinite cardinal  $\kappa$  there exists a rigid system of graphs  $\{X_i\}_{i < 2^\kappa}$ , each graph of cardinality  $\kappa$ . Let  $M_i = G_{fin} X_i$ .*

## Generalized pure subgroups

If  $\kappa$  is an infinite cardinal, a subgroup  $N$  of  $M$  is said to be  $\kappa$ -pure if  $N$  is a direct summand of every subgroup  $N'$  such that  $N \subseteq N' \subseteq M$  and  $|N'/N| < \kappa$ .

### Theorem (Megibben, 1972)

*For every infinite cardinal  $\kappa$  there exists a group  $M$  containing a  $\kappa$ -pure subgroup  $N$  which is not  $\kappa^+$ -pure.*

### Alternative proof.

*Let  $X_\alpha$  be the graph representing the order relation of the ordinal  $\alpha$ . Let  $N = G_{fin} X_\kappa$  and  $M = G_{fin}(X_\kappa \vee X_{\kappa+1})$ .*

## A class of groups

### Theorem

*There exists a class of groups  $M_\alpha$  indexed by all ordinals  $\alpha$  such that for  $\alpha < \beta$  we have  $\text{Hom}(M_\beta, M_\alpha) = 0$ .*

### Proof.

*Let  $X_\alpha$  be the graph representing the order relation of  $\alpha$ .*

*Let  $M_\alpha = GX_\alpha$ .*

The following are equivalent:

1. Negation of Vopěnka's Principle
2. There exists a rigid class of graphs
3. There exists a rigid class of groups

1  $\iff$  2 Definition

3  $\implies$  2 Adámek-Rosický, 1994

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## Chains of group localizations

For any ordinal  $\lambda$  there exists a chain of groups  $M_\alpha$ ,  $\alpha < \lambda$  such that the inclusions  $M_\alpha \subseteq M_\beta$  are localizations for  $\alpha < \beta < \lambda$ .  
That means they induce isomorphisms

$$\mathrm{Hom}(M_\beta, M_\beta) \cong \mathrm{Hom}(M_\alpha, M_\beta)$$

The following are equivalent:

1. Negation of Vopěnka's Principle
2.  $\mathcal{Ord}$  fully embeds into  $\mathcal{Graphs}$
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## Three definitions of a localization $L : \mathcal{C} \rightarrow \mathcal{C}$

1.  $L$  is a left adjoint of some inclusion functor  $\mathcal{D} \subseteq \mathcal{C}$ .
2.  $L$  is a functor with coaugmentation  $\eta : Id \rightarrow L$  such that  $\eta_{LX} = L\eta_X : LX \rightarrow LLX$  is an isomorphism
3. (HPS)
  - ▶  $L\eta : LX \rightarrow LLX$  is an isomorphism for all  $X$
  - ▶ For all  $X$  and  $Y$  the morphism  $\eta_X$  induces a bijection

$$\text{Hom}(LX, LY) \rightarrow \text{Hom}(X, LY)$$

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Localizations may be viewed as projections

onto the class of local objects  $\mathcal{D} = \{Z \mid \eta_Z : Z \xrightarrow{\cong} LZ\}$

along the class of  $L$ -equivalences  $\mathcal{E} = \{f \mid Lf \text{ is an isomorphism}\}$

For every

$f : A \rightarrow B$  in  $\mathcal{E}$ , an  $L$ -equivalence and  
 $Z$  in  $\mathcal{D}$ , an  $L$ -local object

we have:

$$f^* : \mathrm{Hom}(B, Z) \xrightarrow{\cong} \mathrm{Hom}(A, Z)$$

## Orthogonality pairs

We say that  $f : A \rightarrow B$  is orthogonal to  $Z$ , and write  $f \perp Z$ , if

$$f^* : \text{Hom}(B, Z) \xrightarrow{\cong} \text{Hom}(A, Z)$$

A pair  $(\mathcal{E}, \mathcal{D})$  is an orthogonality pair if  $\mathcal{E} = \mathcal{D}^\perp$  and  $\mathcal{D} = \mathcal{E}^\perp$ .  
 A localization always yields an orthogonality pair.

---

Whether every orthogonality pair is associated with a localization depends on set theory

in *Graphs*:

- ▶ NO is consistent with ZFC
- ▶ weak Vopěnka's principle is equivalent to YES

A category  $\mathcal{D}$  is reflective if  $(\mathcal{D}^\perp, \mathcal{D})$  is associated with some localization.

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## Theorem (Isbell problem, 1966)

*The following are equivalent:*

- 1. every closed under limits full subcategory of  $\mathcal{G}raphs$  is reflective (weak Vopěnka's Principle)*
- 2. every closed under limits full subcategory of  $\mathcal{A}b$  is reflective*

1  $\implies$  2 was proved by Adámek and Rosický (1994)

2  $\implies$  1 follows from properties of  $G$

A class  $\mathcal{D}$  is called a small-orthogonality class if it is of the form  $\mathcal{D} = \{f\}^\perp$  for some morphism  $f$ .

## Theorem

*The following are equivalent:*

- every closed under limits full subcategory of  $\mathcal{G}raphs$  is a small-orthogonality class (Vopěnka's Principle)*
- every closed under limits full subcategory of  $\mathcal{A}b$  is a small-orthogonality class*

1  $\implies$  2 was proved by Adámek and Rosický (1994)

2  $\implies$  1 follows from properties of  $G$

## Stable homotopy theory

### Definition (Hovey-Palmieri-Strickland)

An exact localization in a stable homotopy category  $\mathcal{C}$  is a functor  $L$  from  $\mathcal{C}$  to itself with a natural transformation  $\eta : Id \rightarrow L$  such that:

1.  $L\eta : LX \rightarrow LLX$  is an isomorphism for all  $X$
2. For all  $X$  and  $Y$  the morphism  $\eta_X$  induces an isomorphism

$$[LX, LY] \rightarrow [X, LY]$$

3. (HPS) If  $LX = 0$  then  $L(X \wedge Y) = 0$  for all  $Y$

If  $L$  satisfies 1 – 2 only we call it a localization.

In the homotopy category of spectra  $\mathcal{S}$  condition 3 means that  $L$  commutes with the suspension and preserves exact triangles.

Let  $L : \mathcal{S} \rightarrow \mathcal{S}$  be a localization in the homotopy category of spectra. Let  $f : A \rightarrow B$ ,  $Lf$  be an equivalence and  $Z \cong LZ$ . The map of the function spectra

$$f^* : F(B, Z) \rightarrow F(A, Z)$$

induces isomorphisms of the homotopy groups in the nonnegative degrees. If  $L$  is a exact then  $f^*$  induces isomorphisms of all the homotopy groups.

The corresponding notions of orthogonality between  $f$  and  $Z$  lead to orthogonality pairs and exact orthogonality pairs.

## Theorem (Casacuberta-Gutiérrez-Rosický, 2011 preprint)

*Assuming Vopěnka's Principle every class in  $\mathcal{S}$ , closed under homotopy limits, is reflective.*

## Theorem

*Assuming negation of weak Vopěnka's Principle there exists an orthogonality pair  $(\mathcal{E}, \mathcal{D})$  in  $\mathcal{S}$  such that  $\mathcal{D}$  is not reflective (it is closed under homotopy limits).*

These theorems come close to answering the HPS question about existence of exact localizations.

Note that Vopěnka's and weak Vopěnka's Principles are equivalent to their analogues in many categories.

In homotopy category  $Ho$ :

- ▶  $\mathcal{G}raphs \implies Ho$  (Casacuberta-Scevenels-Smith, 2005)
- ▶  $Ho \implies \mathcal{G}raphs$  (AP, 2010)

## Theorem

*Assuming negation of weak Vopěnka's Principle there exists an orthogonality pair in  $\mathcal{S}$  which is not associated with a localization.*

### Proof. (hint)

*Negation of weak Vopěnka's Principle implies existence of an orthogonality pair  $(\mathcal{E}, \mathcal{D})$  in  $\mathcal{A}b$  which is not associated with any localization.*

*We have a full embedding  $\mathcal{A}b \rightarrow \mathcal{S}$ ,  $A \mapsto HA$  where  $(HA)_n = K(A, n)$ .*

$$\pi_n(F(HA, HB)) = [HA, \Sigma^{-n}HB] = \begin{cases} \text{Hom}(A, B) & n = 0 \\ \text{Ext}(A, B) & n = -1 \\ 0 & n > 0 \end{cases}$$

*The embedding preserves orthogonality hence  $(H\mathcal{E}, H\mathcal{D})$  extends to an orthogonality pair  $(H\mathcal{D}^\perp, H\mathcal{D}^{\perp\perp})$  in  $\mathcal{S}$ .*

## Questions

- ▶ Are the following statements equivalent?
  1. weak Vopěnka's Principle
  2. every exact orthogonality pair in  $\mathcal{S}$  is associated with a localization
- ▶ Does  $GX \cong GY$  imply  $X \cong Y$ ?
  - ▶ Yes if  $X$  or  $Y$  is finite
  - ▶  $GX \cong GY$  implies  $|X| = |Y|$
- ▶ Is there an analogue of the functor  $G$  such that  $GX$  has finite rank for finite  $X$ ?
- ▶ Can we modify the construction of  $G$  so that  $GX$  is an  $E$ -ring for rigid graphs  $X$ ?  
More generally: is it possible to reprove some results, obtained by the Shelah Black Box, using methods of category theory.

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## Properties of $G : \mathcal{G}raphs \rightarrow Ab$

1.  $\mathbb{Z}[\text{Hom}_{\mathcal{G}raphs}(X, Y)] \xrightarrow{\cong} \text{Hom}(GX, GY)$
2.  $f \perp Z$  if and only if  $Gf \perp GZ$
3. the cardinality of  $GX$  is at least the continuum
4.  $G$  preserves countably directed colimits
5.  $G$  preserves monomorphisms
6. for every  $a \in GX$  there exists a countable subgraph  $C \subseteq X$  such that  $a \in GC$
7.  $G$  does not preserve products.