IMUB Institut de Matemàtica

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No. 411
February 2009

# Mathematics Preprint Series

UNIVERSITAT DE BARCELONA

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# Picard-Vessiot Theory and Jacobian Problem

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In this paper we establish the equivalence of the two existing definitions of Picard-Vessiot extension of partial differential fields and we give an equivalent statement of the Jacobian conjecture in terms of Picard-Vessiot extensions.

We denote as usual by $K\langle S\rangle$ the differential field differentially generated by the set $S$ and by $C_K$ the field of constants of a differential field $K$.

Kolchin gave in [K] a definition of Picard-Vessiot extension for partial differential fields which is a generalization of the definition of Picard-Vessiot extension associated to a linear differential operator defined over an ordinary differential field. To a partial differential field extension $L|K$, with derivations $\partial_1, \dots, \partial_m$, he associates an ordinary differential field extension $L_D|K_D$, where $K_D := K\langle u_1, \dots, u_m\rangle$, $L_D := L\langle u_1, \dots, u_m\rangle$, with $u_1, \dots, u_m$ independent differential indeterminates, and the derivation is $D := u_1\partial_1 + \dots + u_m\partial_m$. Kolchin defines $L|K$ to be a Picard-Vessiot extension when it satisfies condition **1.** in Theorem 1 below and $C_L = C_K$. Following Kolchin, for elements $\eta_1, \dots, \eta_n$ in a differential field $L$ and $\theta_1, \dots \theta_n$ differential operators defined on $L$, we denote by $W_{\theta_1,\dots\theta_n}(\eta_1, \dots, \eta_n)$ the wronskian determinant $\det((\theta_j(\eta_i))_{1\le i,j\le n})$. Kolchin proves that $L|K$ is Picard-Vessiot if and only if $L_D|K_D$ is Picard-Vessiot and proves that the differential Galois groups $G_{diff}(L|K)$ and $G_{diff}(L_D|K_D)$ are isomorphic. In [P-S] Picard-Vessiot theory for partial differential fields is sketched in terms of differential systems. A Picard-Vessiot extension $L|K$ is there defined by condition **3.** in Theorem 1 below and $C_L = C_K$. The theory in all detail has been worked out by Heiderich in [H]. The next theorem establishes the equivalence of the two definitions.

**Theorem 1** *Let $L|K$ be a partial differential field extension, with derivations $\partial_1, \dots, \partial_m$, such that $C_L = C_K$. The following conditions are equivalent.*

**1.** $L = K\langle \eta_1, \dots, \eta_n\rangle$*, where $\eta_1, \dots, \eta_n$ are linearly independent over constants and*

[1]Both authors acknowledge support by European Network MRTN-CT-2006-035495, by grant MTM2006-04895, Spanish Ministry of Education and by grant N20103831/3261, Polish Ministry of Science and Higher Education.

$$\frac{W_{\theta_1,\ldots\theta_n}(\eta_1,\ldots,\eta_n)}{W_{\theta_{01},\ldots\theta_{0n}}(\eta_1,\ldots,\eta_n)} \in K$$

*for all n-tuples $\theta_1,\ldots\theta_n$ of order $\leq n$ of the semi-group of differential operators generated by $\partial_1,\ldots,\partial_m$ and some fixed $\theta_{01},\ldots\theta_{0n}$ such that $W_{\theta_{01},\ldots\theta_{0n}}(\eta_1,\ldots,\eta_n) \neq 0$.*

**2.** *(i) $L = K\langle E\rangle$, where $E \subset L$ is a finite dimensional $C_K$-vector space.*

*(ii) there exists a group $G$ of differential automorphisms of $L$ such that $E$ is $G$-stable and $L^G = K$.*

**3.** *$L = K\langle\{y_{ij}\}_{1\leq i,j\leq r}\rangle$, where $y = (y_{ij})_{1\leq i,j\leq r}$ is a fundamental matrix of a differential system $\partial_k Y = A_k Y$, with $A_k \in \mathrm{M}_{r\times r}(K)$, $1 \leq k \leq m$, i.e. $y$ is invertible and satisfies $\partial_k y = A_k y$, for $1 \leq k \leq m$.*

*Proof.*

**1. ⇒ 2.**

Let $E$ be the $C_K$-vector space generated by $\eta_1,\ldots,\eta_n$. Then condition **2.**(i) is clearly fulfilled. Let $G$ be the full group $G_{diff}(L|K)$ of differential $K$-automorphisms of $L$. We have $L^G = K$ ([K] Theorem 2). It remains to prove that for all $\sigma \in G$, we have $\sigma\eta_j = \sum c_{ij}\eta_i$ for some $c_{ij} \in C_K$. This comes from the fact that $L_D|K_D$ is a Picard-Vessiot extension of ordinary differential fields, $L_D = K_D\langle\eta_1,\ldots,\eta_n\rangle$ and $G_{diff}(L_D|K_D) \simeq G_{diff}(L|K)$ (see [K]).

**2. ⇒ 3.**

Let $y_1,\ldots,y_r$ be a $C_K$-basis of $E$. As $y_1,\ldots,y_r$ are linearly independent over constants, there exist differential operators $\theta_{01},\ldots,\theta_{0r}$ such that $W_{\theta_{01},\ldots\theta_{0r}}(y_1,\ldots,y_r) \neq 0$. Let $y$ be the matrix $(\theta_{0j}(y_i))_{1\leq i,j\leq r}$. Then $y$ is invertible and $L$ is differentially generated over $K$ by the entries of $y$. Let us see that $\partial_l(y)y^{-1}$ is a matrix with entries in $K$, for $1 \leq l \leq m$. As $E$ is $G$-invariant, we have, for all $\sigma \in G$, $\sigma(y_j) = \sum_{i=1}^r c_{ij}y_i$, for some $c_{ij}$ in $C_K$. Then $\sigma(\theta_{0k}(y_j)) = \sum_{i=1}^r c_{ij}\theta_{0k}(y_i)$, for $1 \leq j,k \leq r$. We obtain then $\sigma(y) = yC$, where $C = (c_{ij})$ is a matrix with entries in $C_K$. Then, as $\sigma$ is a differential automorphism, we have $\sigma((\partial_l y)y^{-1}) = \sigma(\partial_l y)(\sigma(y^{-1}) = \partial_l(\sigma(y))\sigma(y)^{-1} = \partial_l(yC)(yC)^{-1} = \partial_l(y)y^{-1}$. As this yields for all $\sigma$ in $G$, we obtain $\partial_l(y)y^{-1} \in \mathrm{M}_{r\times r}(K), 1 \leq l \leq m$.

**3. ⇒ 2.**

Let $E$ be the $C_K$-vector space generated by the entries $y_{ij}$ of the fundamental matrix. Then $L = K\langle E\rangle$. We take $G = G_{diff}(L|K)$. We have $L^G = K$ by the Fundamental Theorem of Picard-Vessiot theory. We prove that $G(E) \subset E$. For $\sigma \in G$, we have $\partial_l(\sigma y) = \sigma(\partial_l y) = \sigma(A_l y) = A_l\sigma(y)$. So, $\sigma y$ is a fundamental matrix of the differential system $\partial_k Y = A_k Y$, hence $\sigma y = yC$, with $C \in \mathrm{M}_{r\times r}(C_K)$. So, $\sigma y_{ij} = \sum_{k=1}^r c_{kj}y_{ik} \in E$, for $1 \leq i,j \leq r$.

**2. ⇒ 1.**

Let $\eta_1,\ldots,\eta_n$ be a basis of the $C_K$-vector space $E$. Then $L = K\langle\eta_1,\ldots,\eta_n\rangle$. As the elements $\eta_1,\ldots,\eta_n$ are linearly independent over constants, we have $W_{\theta_{01},\ldots\theta_{0n}}(\eta_1,\ldots,\eta_n) \neq 0$, for some $\theta_{01},\ldots\theta_{0n}$. Now, if $\sigma \in G$, $\sigma(\eta_j) = \sum_{i=1}^n c_{ij}\eta_i$, as $E$ is $G$-invariant. Then

$$W_{\theta_1,\dots\theta_n}(\sigma\eta_1,\dots,\sigma\eta_n) = \det(C) W_{\theta_1,\dots\theta_n}(\eta_1,\dots,\eta_n),$$

with $C = (c_{ij})$, for all $\sigma \in G$ and every $n$-tuple $\theta_1,\dots\theta_n$. Hence the quotient $\dfrac{W_{\theta_1,\dots\theta_n}(\eta_1,\dots,\eta_n)}{W_{\theta_{01},\dots\theta_{0n}}(\eta_1,\dots,\eta_n)}$ is fixed by the action of $G$ and is then in $K$.□

**Remark 1** In the ordinary case, if condition **1.** in Theorem 1 is fulfilled, we have $\det((\eta_i^{(j)})_{1\le i\le n, 0\le j\le n-1}) \neq 0$ and $\{\eta_1,\dots,\eta_n\}$ is a fundamental set of solutions of the linear differential equation of order $n$ with coefficients in $K$

$$\begin{vmatrix} Y & \eta_1 & \dots & \eta_n \\ Y' & \eta_1' & \dots & \eta_n' \\ & & \vdots & \\ Y^{(n)} & \eta_1^{(n)} & \dots & \eta_n^{(n)} \end{vmatrix} = 0.$$

**Remark 2** In the ordinary case, the equivalence between **1.** and **2.** is given by Magid in [M].

**Remark 3** In the ordinary case, we can associate to a differential system given by an $n \times n$ matrix an equivalent system corresponding to a linear differential equation **of order** $n$, by means of cyclic vectors.

We give now an equivalent statement of the Jacobian conjecture in terms of Picard-Vessiot extensions. Let $\mathbb{K}$ be a field of characteristic 0 and $F := (F_1, F_2, \dots, F_n)$ be a polynomial function defined on the field of rational functions $\mathbb{K}(x_1, x_2, \dots, x_n)$. We consider the Jacobian matrix

$$J := \frac{\partial(F_1, F_2, \dots, F_n)}{\partial(x_1, x_2, \dots, x_n)}$$

and assume that the Jacobian determinant $|J|$ is a non zero constant. The field $\mathbb{K}(x_1, x_2, \dots, x_n)$ is then a differential field with the Nambu derivations defined by

$$(1) \qquad \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{pmatrix} = (J^{-1})^t \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \vdots \\ \partial/\partial x_n \end{pmatrix}$$

(see [N]). It is easy to check that the field $\mathbb{K}(F_1, F_2, \dots, F_n)$ is stable under $\{\delta_1, \delta_2, \dots, \delta_n\}$.

**Theorem 2** *We assume that $\mathbb{K}$ is algebraically closed. Let $F := (F_1, F_2, \ldots, F_n)$ be a polynomial function defined on $\mathbb{K}(x_1, x_2, \ldots, x_n)$ such that $|J| \in \mathbb{K} \setminus \{0\}$. The following conditions are equivalent.*

**1.** *The function $F$ is invertible and its inverse is a polynomial function.*

**2.** *The differential field $(\mathbb{K}(x_1, x_2, \ldots, x_n), \{\delta_1, \delta_2, \ldots, \delta_n\})$ is a Picard-Vessiot extension of $\mathbb{K}$.*

**3.** *The finite field extension $\mathbb{K}(x_1, x_2, \ldots, x_n)|\mathbb{K}(F_1, F_2, \ldots, F_n)$ is a Galois extension.*

**4.** *The elements $x_1, \ldots, x_n$ satisfy $W_{\delta_1 \ldots \delta_n}(x_1, \ldots, x_n) \neq 0$ and*

$$\frac{W_{\theta_1, \ldots \theta_n}(x_1, \ldots, x_n)}{W_{\delta_1 \ldots \delta_n}(x_1, \ldots, x_n)} \in \mathbb{K}(F_1, F_2, \ldots, F_n)$$

*for all $n$-tuples $\theta_1, \ldots, \theta_n$ of order $\leq n+1$ of the semi-group of differential operators generated by $\delta_1, \ldots, \delta_n$.*

*Proof.*
**1.** $\Rightarrow$ **2.** If $F$ is invertible, we have $\mathbb{K}(F_1, F_2, \ldots, F_n) = \mathbb{K}(x_1, x_2, \ldots, x_n)$. By (1), we have $\delta_i(F_j) = \delta_{ij}$, so the conditions in Theorem 1 **1.** are satisfied if we take $1, F_1, \ldots, F_n$ as the extension generators and $1, \delta_1, \ldots, \delta_n$ as the operators $\theta_{0j}$.
**2.** $\Rightarrow$ **3.** As $(\mathbb{K}(x_1, x_2, \ldots, x_n), \{\delta_1, \delta_2, \ldots, \delta_n\})$ is a Picard-Vessiot extension of $\mathbb{K}$ and $\mathbb{K}(F_1, F_2, \ldots, F_n)$ is an intermediate differential field, we have that $\mathbb{K}(x_1, x_2, \ldots, x_n)$ is a Picard-Vessiot extension of $\mathbb{K}(F_1, F_2, \ldots, F_n)$. As the constant field is the algebraically closed field $\mathbb{K}$, the extension $\mathbb{K}(x_1, x_2, \ldots, x_n)|\mathbb{K}(F_1, F_2, \ldots, F_n)$ is a Galois extension.
**3.** $\Rightarrow$ **1.** This is a Theorem of Campbell (see [C] or [E]).
**1.** $\Rightarrow$ **4.** By (1), we have $(\delta_j(x_i)) = J^{-1}$, so $W_{\delta_1 \ldots \delta_n}(x_1, \ldots, x_n) = \det(\delta_j(x_i)) = |J|^{-1} \in \mathbb{K} \setminus \{0\}$. Now $W_{\theta_1, \ldots \theta_n}(x_1, \ldots, x_n)/W_{\delta_1 \ldots \delta_n}(x_1, \ldots, x_n)$ is clearly in $\mathbb{K}(x_1, \ldots, x_n)$, so in $\mathbb{K}(F_1, \ldots, F_n)$.
**4.** $\Rightarrow$ **3.** Condition **1.** in Theorem 1 is satisfied by the elements $1, x_1, \ldots, x_n$ and the derivations $1, \delta_1, \ldots, \delta_n$, so the extension $\mathbb{K}(x_1, x_2, \ldots, x_n)|\mathbb{K}(F_1, F_2, \ldots, F_n)$ is a Picard-Vessiot extension, hence a Galois extension.

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