

A Large Deviation Principle in Hölder Norm for Multiple Fractional Integrals

MARTA SANZ-SOLÉ^(*)
marta.sanz@ub.edu

and

IVÁN TORRECILLA-TARANTINO^(*)
itorrecilla@ub.edu

Facultat de Matemàtiques
Universitat de Barcelona
Gran Via 585
08007 Barcelona, Spain

Abstract: For a fractional Brownian motion B^H with Hurst parameter $H \in]\frac{1}{4}, \frac{1}{2}[\cup]\frac{1}{2}, 1[$, multiple indefinite integrals on a simplex are constructed and the regularity of their sample paths are studied. Then, it is proved that the family of probability laws of the processes obtained by replacing B^H by $\varepsilon^{\frac{1}{2}}B^H$ satisfies a large deviation principle in Hölder norm. The definition of the multiple integrals relies upon a representation of the fractional Brownian motion in terms of a stochastic integral with respect to a standard Brownian motion. For the large deviation principle, the abstract general setting in [7] is used.

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1 Introduction

In this paper, we consider stochastic processes $X = \{X_t, t \in [0, T]\}$ given by indefinite multiple integrals on the n -dimensional simplex $\{(\theta_1, \dots, \theta_n) \in \mathbb{R}_+^n : 0 \leq \theta_1 \leq \dots \leq \theta_n \leq t\}$ with respect to a fractional Brownian motion (fBm). The integrands are allowed to depend also on the parameter t . Under suitable assumptions on the integrands, depending whether the Hurst parameter H belongs to $] \frac{1}{2}, 1[$ or $] \frac{1}{4}, \frac{1}{2}[$, we prove Hölder continuity of the sample paths, a.s.. Then we establish a large deviation principle (LDP) in Hölder norm for the family of laws of $\varepsilon^{\frac{n}{2}} X$.

For the standard Brownian motion (sBm), a similar question has been addressed in [11]. The authors consider different assumptions on the integrands ensuring a.s. continuity of the sample paths of the integrals; then they prove large deviations principles in the space of continuous functions endowed with the supremum norm. Geometric rough paths based on processes with γ -Hölder continuous sample paths give rise to random vectors whose components are multiple Stratonovich integrals up to order $[\frac{1}{\gamma}]$, where $[\cdot]$ denotes the integer value. A large deviation principle for the rough path lying above the standard Brownian motion has been proved in [8]. The norm under consideration is the p -variation norm used in the rough path analysis (see for instance [10]). The higher order of the multiple stochastic integrals involved is in this example $n = 2$. For the fractional Brownian motion with Hurst parameter $H \in] \frac{1}{4}, \frac{1}{2}[\cup] \frac{1}{2}, 1[$, a similar result has been proved in [12]. We notice that the non trivial part of it corresponds to the values $H \in] \frac{1}{4}, \frac{1}{2}[$ and that one needs to deal with multiple stochastic integrals up to order $n = 3$.

This paper is motivated mainly by [11] and [12] in the following sense: As in [11], we want to consider multiple indefinite integrals of any order and on the other hand, we wish to deal with sharper norms, like Hölder norm, and with the fBm.

The main body of the paper is devoted to the construction of the indefinite multiple integral on a simplex with respect to the fBm, and the study of its sample paths. The corresponding results are gathered in Section 3. Starting from the results and ideas in [1] for $n = 1$, by means of a recursive argument, we are able to give a meaning to the multiple integral with respect to the fBm as a multiple integral with respect to the sBm. For this, we identify the kernels corresponding to increments in time of such integrals. With suitable assumptions, we prove that these kernels define continuous operators on the space of the integrands taking values on spaces of Hölder-continuous functionals (see (13) and (21)). By means of the hypercontractivity property of Gaussian chaos, the Hölder continuity is transferred to the sample paths of the integrals.

We should mention the fractional calculus approach (see for instance [18]) to multiple definite integrals with respect to the fBm given in [16], and to indefinite integrals of progressively measurable processes with respect to the fBm -including sample path properties- in [4]. In contrast, as we have mentioned before, our approach follows [1] (see also [6] and [5]). It is based on anticipating integrals of Skorohod type; thus, on techniques from Malliavin calculus.

Once we have identified the functional spaces where our fBm functionals live, we can study what LDP they do satisfy. In [7], a LDP for random vectors in a Banach-

valued homogeneous Wiener chaos of any order n is established. The elegant proof relies upon isoperimetric methods. This provides the suitable framework for our study. In fact, in Section 4 we first identify the abstract Wiener space associated with the fBm as a Gaussian process with Hölder continuous paths. Then we notice that the space of γ -Hölder continuous functions can be embedded in a separable Banach space (see the first part of Section 4 for some details and references). With this and the results of Section 3, we see that the indefinite multiple integrals with respect to the fBm are Banach-valued random vectors in a Wiener chaos. Therefore, the results of [7] can be applied. A similar approach could be used for the sBm to obtain the LDP stated in [11] and very likely with sharper norms.

2 Preliminaries and notation

We start the article with this section devoted to fix the notation and recall some known facts that will be intensively used throughout the paper, refereeing to [15] and the references herein for additional details.

Let $B^H = \{B_t^H, t \in [0, T]\}$ be a fBm with Hurst parameter $H \in]0, 1[$. The process B^H can be represented in terms of a stochastic integral with respect to a sBm $W = \{W_t, t \in [0, T]\}$ as follows:

$$B_t^H = \int_0^t K_H(t, \theta) dW_\theta, \quad (1)$$

where dW_θ denotes the Itô differential and

$$K_H(t, \theta) := c_H \left\{ (t - \theta)^{H - \frac{1}{2}} + \left(\frac{1}{2} - H \right) \int_\theta^t (u - \theta)^{H - \frac{3}{2}} \left(1 - \left(\frac{\theta}{u} \right)^{\frac{1}{2} - H} \right) du \right\}, \quad (2)$$

and c_H is some positive constant depending on H . Then

$$\frac{\partial K_H}{\partial t}(t, \theta) = c_H \left(H - \frac{1}{2} \right) \left(\frac{\theta}{t} \right)^{\frac{1}{2} - H} (t - \theta)^{H - \frac{3}{2}}. \quad (3)$$

Thus, for $H \in]0, \frac{1}{2}[$, the derivative $\frac{\partial K_H}{\partial t}(t, \theta)$ is negative and moreover,

$$\left| \frac{\partial K_H}{\partial t}(t, \theta) \right| \leq C_H |t - \theta|^{H - \frac{3}{2}}. \quad (4)$$

Notice that for $H > \frac{1}{2}$ the kernel $K_H(t, \theta)$ is regular and for $H < \frac{1}{2}$ it is singular. For any $h \in L^2([0, T])$ we define the operator K_H by

$$(K_H h)(t) = \int_0^t K_H(t, \theta) h(\theta) d\theta.$$

Let \mathcal{E} be the set of step functions on $[0, T]$. We define the $L^2([0, T])$ -valued linear operator K_H^* on \mathcal{E} by

$$(K_H^* \varphi)(\theta) = \varphi(\theta) K_H(T, \theta) + \int_\theta^T [\varphi(r) - \varphi(\theta)] K_H(dr, \theta). \quad (5)$$

The operator K_H^* is the adjoint of K_H in the following sense:

For any function $\varphi \in \mathcal{E}$ and $h \in L^2([0, T])$, one has

$$\int_0^T (K_H^* \varphi)(r) h(r) dr = \int_0^T \varphi(r) (K_H h)(dr). \quad (6)$$

Replacing $h(s)ds$ by dW_s , with the same proof of (6) it can be checked that for any $\varphi \in \mathcal{E}$ the element $B^H(\varphi) := \int_0^T \varphi(\theta) dB_\theta^H$ of the first Gaussian chaos associated with the fBm can be written as

$$B^H(\varphi) = \int_0^T (K_H^* \varphi)(\theta) dW_\theta.$$

Let $t \in [0, T]$. Then

$$\begin{aligned} (K_{H,t}^* \varphi)(\theta) &:= (K_H^*(\varphi \mathbb{1}_{[0,t]}))(\theta) \\ &= \varphi(\theta) K_H(t, \theta) + \int_\theta^t [\varphi(r) - \varphi(\theta)] K_H(dr, \theta). \end{aligned} \quad (7)$$

Notice that $K_{H,T}^* = K_H^*$.

Thus,

$$B^H(\varphi \mathbb{1}_{[0,t]}) = \int_0^t \varphi(\theta) dB_\theta^H = \int_0^t (K_{H,t}^* \varphi)(\theta) dW_\theta.$$

For $H \in]\frac{1}{2}, 1[$ and $\varphi \in \mathcal{E}$, the kernel $K_{H,t}^*$ has the simple expression

$$(K_{H,t}^* \varphi)(\theta) = \int_\theta^t \varphi(r) K_H(dr, \theta). \quad (8)$$

For this same range of H , denote by $|\mathcal{H}^H|$ the linear space consisting of measurable functions φ defined on $[0, T]$ such that

$$\begin{aligned} \|\varphi\|_{|\mathcal{H}^H|}^2 &:= \int_0^T \left(\int_\theta^T |\varphi_r| K_H(dr, \theta) \right)^2 d\theta \\ &= \alpha_H \int_0^T \int_0^T |\varphi_r| |\varphi_\xi| |r - \xi|^{2H-2} dr d\xi < \infty, \end{aligned}$$

where $\alpha_H = H(2H - 1)$. The space $|\mathcal{H}^H|$ endowed with the norm $\|\cdot\|_{|\mathcal{H}^H|}$, is a Banach space. Hölder's inequality with exponent $q = \frac{1}{H}$ and the Hardy-Littlewood-Sobolev inequality (see for instance, [19], page 354) yields

$$\|\varphi\|_{|\mathcal{H}^H|} \leq b_H \|\varphi\|_{L^{\frac{1}{H}}([0,T])}. \quad (9)$$

For $H \in]0, \frac{1}{2}[$, we introduce the seminorm on \mathcal{E}

$$\|\varphi\|_{K_H}^2 := \int_0^T \varphi_\theta^2 K_H(T, \theta)^2 d\theta + \int_0^T \left(\int_\theta^T |\varphi_r - \varphi_\theta| |K_H(dr, \theta)| \right)^2 d\theta.$$

By \mathcal{H}_{K_H} , we denote the completion of \mathcal{E} with respect to this seminorm. It consists of functions φ defined on $[0, T]$ such that $\|\varphi\|_{K_H}^2 < \infty$.

For any $t \in [0, T]$, set

$$\Lambda_t^{(n)} = \{(\theta_1, \dots, \theta_n) \in \mathbb{R}_+^n : 0 \leq \theta_1 \leq \dots \leq \theta_n \leq t\}.$$

Throughout the paper, we denote by $\mathbb{H}^\lambda(\Lambda_T^{(n)})$, $\lambda \in]0, 1[$, the space of λ -Hölder continuous functions on the k -cubes contained on $\Lambda_t^{(n)}$, $1 \leq k \leq n$, endowed with the norm

$$\begin{aligned} \|h\|_{\mathbb{H}^\lambda(\Lambda_t^{(n)})} &= \sup_{(\theta_1, \dots, \theta_n) \in \Lambda_t^{(n)}} |h(\theta_1, \dots, \theta_n)| \\ &+ \sum_{k=1}^n \sum_{i_1 < \dots < i_k=1}^n \sup_{\substack{(\theta_1, \dots, \theta_n) \in \Lambda_t^{(n)} \\ 0 \leq \theta_{i_1} < r_{i_1} \leq \theta_{i_2} < r_{i_2} \leq \dots \leq \theta_{i_k} < r_{i_k} \leq t}} \frac{|\Delta^{i_1, \dots, i_k} h(\theta_1, \dots, \theta_n; r_{i_1}, \dots, r_{i_k})|}{\prod_{j=1}^k |r_{i_j} - \theta_{i_j}|^\lambda}, \end{aligned}$$

where for $n = 1$, $\Delta^1 h(\theta, r) = h(r) - h(\theta)$, and for $n \geq 2$,

$$\begin{aligned} \Delta^{i_1} h(\theta_1, \dots, \theta_n; r_{i_1}) &= h(\theta_1, \dots, r_{i_1}, \dots, \theta_n) - h(\theta_1, \dots, \theta_{i_1}, \dots, \theta_n), \\ \Delta^{i_1, \dots, i_k} h(\theta_1, \dots, \theta_n; r_{i_1}, \dots, r_{i_k}) &= \Delta^{i_1, \dots, i_{k-1}} h(\theta_1, \dots, r_{i_k}, \dots, \theta_n; r_{i_1}, \dots, r_{i_{k-1}}) \\ &\quad - \Delta^{i_1, \dots, i_{k-1}} h(\theta_1, \dots, \theta_{i_k}, \dots, \theta_n; r_{i_1}, \dots, r_{i_{k-1}}), \end{aligned}$$

$2 \leq k \leq n$.

Given a Banach space \mathbb{B} , we shall denote by $\mathcal{C}^\lambda([0, T]; \mathbb{B})$ ($\mathcal{C}^\lambda([0, T])$, when $\mathbb{B} = \mathbb{R}$) the space of λ -Hölder continuous functions endowed with the norm

$$\|h\|_{\mathcal{C}^\lambda([0, T]; \mathbb{B})} = \sup_{0 \leq t \leq T} \|h(t)\|_{\mathbb{B}} + \sup_{\substack{0 \leq s, t \leq T \\ s \neq t}} \frac{\|h(t) - h(s)\|_{\mathbb{B}}}{|t - s|^\lambda}.$$

For $n = 1$, the space $\mathbb{H}^\lambda(\Lambda_T^{(1)})$ is usually denoted by $\mathcal{C}^\lambda([0, T])$.

For any $p \in [1, \infty[$, we denote by $\mathbb{D}^{1,p}(\mathbb{B})$ the space of \mathbb{B} -valued random variables Y satisfying

$$\|Y\|_{1,p,\mathbb{B}}^p := \mathbb{E}(\|Y\|_{\mathbb{B}}^p) + \mathbb{E}\left(\int_0^T \|D_\eta Y\|_{\mathbb{B}}^2 d\eta\right)^{\frac{p}{2}} < +\infty.$$

In the next Propositions 2.1 and 2.2, we shall use this notation for $\mathbb{B} = |\mathcal{H}^H|$ and $\mathbb{B} = \mathcal{H}_{K_H}$, respectively.

For $\mathbb{B} = \mathbb{R}$, we write $\mathbb{D}^{1,p}$ instead of $\mathbb{D}^{1,p}(\mathbb{R})$, and this is the usual Sobolev-Watanabe space associated with the Wiener process W .

Set

$$X_t = \int_0^t u_\theta dB_\theta^H, \quad t \in [0, T],$$

where the meaning attached to the stochastic integral is that of [1], that is

$$\int_0^t u_\theta dB_\theta^H := \int_0^t (K_{H,t}^* u)(\theta) dW_\theta.$$

The sample path properties of the multiple stochastic integrals investigated in this paper rely on results telling us what properties on the integrand u imply that the stochastic process $\{X_t, t \in [0, T]\}$ exist and its sample paths are Hölder continuous functions.

An answer is provided by Propositions 3 and 1 of [1], by considering the fractional Brownian motion as Gaussian process and taking in these statements $\alpha = H - \frac{1}{2}$, with $H \in]\frac{1}{2}, 1[$, and $\alpha = \frac{1}{2} - H$, with $H \in]0, \frac{1}{2}[$, respectively. The next Proposition 2.1, gives a slightly different result when $H \in]\frac{1}{2}, 1[$, while Proposition 2.2 is just a quotation of Proposition 1 of [1].

Proposition 2.1 *Let $H \in]\frac{1}{2}, 1[$ and $p \in [2, \infty[$. Consider a stochastic process $u = \{u_t, t \in [0, T]\}$ belonging to $L^{\frac{1}{H}}([0, T]; \mathbb{D}^{1,p})$. Then $u \in \mathbb{D}^{1,p}(|\mathcal{H}^H|)$ and the stochastic integral process $X = \{X_t, t \in [0, T]\}$ is well defined and consists of random variables in $L^p(\Omega)$.*

Furthermore, if $qH > 1$, and $u \in L^q([0, T]; \mathbb{D}^{1,p})$, then

$$\mathbb{E}|X_t - X_s|^p \leq C|t - s|^{p(H-1/q)}.$$

Hence, if $p(H - 1/q) > 1$, the process X has γ -Hölder continuous paths, a.s., with $\gamma \in]0, H - 1/q - 1/p[$.

Proof: By applying twice Minkowski's inequality, we have

$$\begin{aligned} & \mathbb{E} \left(\|u\|_{|\mathcal{H}^H|}^p \right) \\ &= \left\| \int_0^T d\theta \left(\int_\theta^T |u_r| K(dr, \theta) \right)^2 \right\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{p}{2}} \leq \left(\int_0^T d\theta \left\| \int_\theta^T |u_r| K(dr, \theta) \right\|_{L^p(\Omega)}^2 \right)^{\frac{p}{2}} \\ &\leq \left(\int_0^T d\theta \left(\int_\theta^T dr \left\| u_r \frac{\partial K_H}{\partial r}(r, \theta) \right\|_{L^p(\Omega)} \right)^2 \right)^{\frac{p}{2}} \leq \left(\int_0^T d\theta \left(\int_\theta^T \|u_r\|_{1,p} K(dr, \theta) \right)^2 \right)^{\frac{p}{2}}. \end{aligned}$$

By using first Fubini's theorem and then Minkowski's inequality three times, we obtain

$$\begin{aligned} \mathbb{E} \left(\int_0^T \|D_\eta u\|_{|\mathcal{H}^H|}^2 d\eta \right)^{\frac{p}{2}} &= \mathbb{E} \left(\int_0^T d\theta \left\| \int_\theta^T |D_\cdot u_r| K(dr, \theta) \right\|_{L^2([0,T])}^2 \right)^{\frac{p}{2}} \\ &\leq \left\| \int_0^T d\theta \left(\int_\theta^T \|D_\cdot u_r\|_{L^2([0,T])} K(dr, \theta) \right)^2 \right\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{p}{2}} \\ &\leq \left(\int_0^T d\theta \left\| \int_\theta^T \|D_\cdot u_r\|_{L^2([0,T])} K(dr, \theta) \right\|_{L^p(\Omega)}^2 \right)^{\frac{p}{2}} \\ &\leq \left(\int_0^T d\theta \left(\int_\theta^T \|u_r\|_{1,p} K(dr, \theta) \right)^2 \right)^{\frac{p}{2}}. \end{aligned}$$

Since by (9)

$$\int_0^T \left(\int_\theta^T \|u_r\|_{1,p} K_H(dr, \theta) \right)^2 d\theta \leq b_H^2 \left(\int_0^T \|u_r\|_{1,p}^{\frac{1}{H}} dr \right)^{2H},$$

we obtain that $u \in \mathbb{D}^{1,p}(|\mathcal{H}^H|)$.

For $0 \leq s < t$, following the proof of Proposition 1 in [1], we can write

$$X_t - X_s = \int_0^s \left(\int_s^t u_r K_H(dr, \theta) \right) dW_\theta + \int_s^t \left(\int_\theta^t u_r K_H(dr, \theta) \right) dW_\theta.$$

Thus, Meyer's inequality implies

$$\mathbb{E} |X_t - X_s|^p \leq C(p) \left(S_1^2 + S_2^2 \right)^{\frac{p}{2}},$$

with

$$S_1 = \left\| \mathbb{1}_{]0,s[} \int_s^t \|u_r\|_{1,p} K_H(dr, \cdot) \right\|_{L^2([0,t])},$$

$$S_2 = \left\| \mathbb{1}_{]s,t[} \int_s^t \|u_r\|_{1,p} K_H(dr, \cdot) \right\|_{L^2([0,t])}.$$

By comparing the domains of integration in the terms S_1 and S_2 above with that of $\left\| \|u_r\|_{1,p} \mathbb{1}_{]s,t[} \right\|_{|\mathcal{H}^H|}$ we see that

$$S_1^2 + S_2^2 \leq \left\| \|u_r\|_{1,p} \mathbb{1}_{]s,t[} \right\|_{|\mathcal{H}^H|}^2.$$

Then, applying (9) we obtain

$$\mathbb{E} |X_t - X_s|^p \leq C(p, H) \left\| \|u_r\|_{1,p} \mathbb{1}_{]s,t[} \right\|_{L^{\frac{1}{H}}([0,T])}^p.$$

Notice that this last integral is finite.

Fix $q \geq 1$ such that $qH > 1$. By applying Hölder's inequality with $p' = qH$, $q' = \frac{qH}{qH-1}$, we reach

$$\begin{aligned} \left\| \|u_r\|_{1,p} \mathbb{1}_{]s,t[} \right\|_{L^{\frac{1}{H}}([0,T])}^p &= \left(\int_s^t \|u_r\|_{1,p}^{\frac{1}{H}} dr \right)^{pH} \\ &\leq (t-s)^{p(H-1/q)} \left(\int_s^t \|u_r\|_{1,p}^q dr \right)^{\frac{p}{q}} \\ &\leq (t-s)^{p(H-1/q)} \|u\|_{L^q([0,T]; \mathbb{D}^{1,p})}^p. \end{aligned}$$

Consequently,

$$\mathbb{E} |X_t - X_s|^p \leq C |t-s|^{p(H-1/q)},$$

and we conclude by applying Kolmogorov's continuity criterion. \square

Proposition 2.2 *Let $H \in]0, \frac{1}{2}[$, $p \in [2, \infty[$. Suppose that the stochastic process $u = \{u_t, t \in [0, T]\}$ belongs to $\mathcal{C}^\lambda([0, T]; \mathbb{D}^{1,p})$ for some $\lambda + H > \frac{1}{2}$. Then u belongs to the space $\mathbb{D}^{1,p}(\mathcal{H}_{K_H})$ and the stochastic integral process $\{X_t, t \in [0, T]\}$ is well defined; it consists of $L^p(\Omega)$ random variables and satisfies $\mathbb{E}|X_t - X_s|^p \leq C|t - s|^{pH}$. Consequently, if $pH > 1$, a.s., the sample paths are γ -Hölder continuous with $\gamma \in]0, H - 1/p[$.*

3 Multiple stochastic integrals of deterministic functions with respect to a fractional Brownian motion

In this section, we study conditions on deterministic functions h defined on $[0, T]^n$ allowing to define the indefinite multiple stochastic integral with respect to a fractional Brownian motion

$$\mathbf{I}_t^{(n),H}(h) = \int_{\Lambda_t^{(n)}} h(\theta_1, \dots, \theta_n) dB_{\theta_1}^H \cdots dB_{\theta_n}^H, \quad t \in [0, T].$$

• **The case $H \in]\frac{1}{2}, 1[$.**

Fix $0 \leq s < t \leq T$. We set

$$\begin{aligned} (K_{H,s,t}^{*,(1)}h)(\theta_1) &= \int_{\theta_1}^t h(r_1) K_H(dr_1, \theta_1) \mathbb{1}_{]s,t[}(\theta_1) \\ &\quad + \int_s^t h(r_1) K_H(dr_1, \theta_1) \mathbb{1}_{]0,s[}(\theta_1) \mathbb{1}_{\{s \neq 0\}}, \end{aligned} \quad (10)$$

and for any integer $n \geq 2$,

$$\begin{aligned} &(K_{H,s,t}^{*,(n)}h)(\theta_1, \dots, \theta_n) \\ &= \int_{\bigvee_{i=1}^n \theta_i}^t (K_{H,r_n}^{*,(n-1)}h(\cdot, r_n))(\theta_1, \dots, \theta_{n-1}) K_H(dr_n, \theta_n) \\ &\quad \times \mathbb{1}_{[0,t]^{n-1}}(\theta_1, \dots, \theta_{n-1}) \mathbb{1}_{]s,t[}(\theta_n) \\ &\quad + \int_{\bigvee_{i=1}^{n-1} \theta_i \vee s}^t (K_{H,r_n}^{*,(n-1)}h(\cdot, r_n))(\theta_1, \dots, \theta_{n-1}) K_H(dr_n, \theta_n) \\ &\quad \times \mathbb{1}_{[0,t]^{n-1}}(\theta_1, \dots, \theta_{n-1}) \mathbb{1}_{]0,s[}(\theta_n) \mathbb{1}_{\{s \neq 0\}}, \end{aligned} \quad (11)$$

where we write $K_{H,\cdot}^{*,(m)}$ for $K_{H,0}^{*,(m)}$.

Proposition 3.1 Fix $H \in]\frac{1}{2}, 1[$ and a natural number $n \geq 1$.

(a) Let $h \in L^{\frac{1}{H}}(\Lambda_T^{(n)})$. Then, for $0 \leq s < t \leq T$,

$$\left\| K_{H,s,t}^{*,(n)} h \right\|_{L^2([0,t]^n)} \leq 2^{n/2} b_H^n \left\| h \mathbb{1}_{]s,t[} \right\|_{L^{\frac{1}{H}}(\Lambda_t^{(n)})}, \quad (12)$$

where b^H is the same constant as in (9). Thus, $K_{H,s,t}^{*,(n)} : L^{\frac{1}{H}}(\Lambda_t^{(n)}) \longrightarrow L^2([0,t]^n)$ defined in (10) and (11) is a linear continuous operator.

(b) If $h \in L^q(\Lambda_T^{(n)})$ with $qH > 1$, then for $0 \leq s < t \leq T$,

$$\left\| K_{H,s,t}^{*,(n)} h \right\|_{L^2([0,t]^n)} \leq 2^{n/2} b_H^n \|h\|_{L^q(\Lambda_t^{(n)})} |t - s|^{H-1/q}. \quad (13)$$

Proof: Let $n = 1$. Owing to (10),

$$\left\| K_{H,s,t}^{*,(1)} h \right\|_{L^2([0,t])} \leq T_1 + T_2,$$

with

$$\begin{aligned} T_1 &= \left\| \mathbb{1}_{]s,t[} \int_{\cdot}^t h(r_1) K_H(dr_1, \cdot) \right\|_{L^2([0,t])}, \\ T_2 &= \left\| \mathbb{1}_{]0,s[} \mathbb{1}_{\{s \neq 0\}} \int_s^t h(r_1) K_H(dr_1, \cdot) \right\|_{L^2([0,t])}. \end{aligned}$$

As in the proof of Proposition 2.1, we see that

$$T_1^2 + T_2^2 \leq \left\| h \mathbb{1}_{]s,t[} \right\|_{|\mathcal{H}^H|}^2. \quad (14)$$

Indeed, it suffices to compare the domains of integration of the terms T_1 and T_2 above with that of $\left\| h \mathbb{1}_{]s,t[} \right\|_{|\mathcal{H}^H|}$. Then, applying (9) we obtain (12) for $n = 1$.

Hölder's inequality with $p' = qH$, $q' = \frac{qH}{qH-1}$, yields

$$\left\| h \mathbb{1}_{]s,t[} \right\|_{L^{\frac{1}{H}}([0,T])} \leq \left\| h \mathbb{1}_{]s,t[} \right\|_{L^q([0,T])} |t - s|^{H-1/q} \leq \|h\|_{L^q([0,T])} |t - s|^{H-1/q}.$$

Hence (13) holds for $n = 1$.

Assume now that (12) holds up to an integer $n' \geq 1$. Let $h \in L^{\frac{1}{H}}(\Lambda_t^{(n'+1)})$. Then $h(\cdot, r_{n'+1}) \in L^{\frac{1}{H}}(\Lambda_{r_{n'+1}}^{(n')})$, for all $r_{n'+1} \in [0, t]$, a.e. and the induction hypothesis yields

$$\left\| K_{H,r_{n'+1}}^{*,(n')} h(\cdot, r_{n'+1}) \right\|_{L^2([0,r_{n'+1}]^{n'})} \leq 2^{n'/2} b_H^{n'} \|h(\cdot, r_{n'+1})\|_{L^{\frac{1}{H}}(\Lambda_{r_{n'+1}}^{(n')})},$$

for all $r_{n'+1} \in [0, t]$, a.e.

From (11) it follows that $\left\| K_{H,s,t}^{*,(n'+1)} h \right\|_{L^2([0,t]^{n'+1})}^2 \leq 2(Q_1 + Q_2)$, with

$$\begin{aligned}
Q_1 &= \int_s^t d\theta_{n'+1} \left\| \int_{\bigvee_{i=1}^{n'+1} \theta_i}^t \left(K_{H,r_{n'+1}}^{*,(n')} h(\cdot, r_{n'+1}) \right) (\cdot) K_H(dr_{n'+1}, \theta_{n'+1}) \right\|_{L^2([0,t]^{n'})}^2, \\
Q_2 &= \int_0^s d\theta_{n'+1} \left\| \int_{\bigvee_{i=1}^{n'} \theta_i \vee s}^t \left(K_{H,r_{n'+1}}^{*,(n')} h(\cdot, r_{n'+1}) \right) (\cdot) K_H(dr_{n'+1}, \theta_{n'+1}) \right\|_{L^2([0,t]^{n'})}^2.
\end{aligned}$$

Minkowski's inequality yields

$$\begin{aligned}
Q_1 &\leq \int_s^t d\theta_{n'+1} \left(\int_{\theta_{n'+1}}^t \left\| K_{H,r_{n'+1}}^{*,(n')} h(\cdot, r_{n'+1}) \right\|_{L^2([0,r_{n'+1}]^{n'})} K_H(dr_{n'+1}, \theta_{n'+1}) \right)^2 \\
&\leq 2^{n'} b_H^{2n'} \int_s^t d\theta_{n'+1} \left(\int_{\theta_{n'+1}}^t \|h(\cdot, r_{n'+1})\|_{L^{\frac{1}{H}}(\Lambda_{r_{n'+1}}^{(n')})} K_H(dr_{n'+1}, \theta_{n'+1}) \right)^2
\end{aligned}$$

and

$$\begin{aligned}
Q_2 &\leq \int_0^s d\theta_{n'+1} \left(\int_s^t \left\| K_{H,r_{n'+1}}^{*,(n')} h(\cdot, r_{n'+1}) \right\|_{L^2([0,r_{n'+1}]^{n'})} K_H(dr_{n'+1}, \theta_{n'+1}) \right)^2 \\
&\leq 2^{n'} b_H^{2n'} \int_0^s d\theta_{n'+1} \left(\int_s^t \|h(\cdot, r_{n'+1})\|_{L^{\frac{1}{H}}(\Lambda_{r_{n'+1}}^{(n')})} K_H(dr_{n'+1}, \theta_{n'+1}) \right)^2.
\end{aligned}$$

Notice that, except for the constant $2^{n'} b_H^{2n'}$, the upper bounds of the terms Q_1, Q_2 coincide with T_1^2 and T_2^2 , respectively, with $h := \|h(\cdot, r_{n'+1})\|_{L^{\frac{1}{H}}(\Lambda_{r_{n'+1}}^{(n')})}$.

Thus, using (14) and (9)

$$\begin{aligned}
\left\| K_{H,s,t}^{*,(n'+1)} h \right\|_{L^2([0,t]^{n'+1})} &\leq 2^{\frac{1}{2}} (Q_1 + Q_2)^{\frac{1}{2}} \\
&\leq 2^{\frac{n'+1}{2}} b_H^{n'} \left\| \|h(\cdot, r_{n'+1})\|_{L^{\frac{1}{H}}(\Lambda_{r_{n'+1}}^{(n')})} \mathbb{1}_{]s,t[} \right\|_{|\mathcal{H}^H|} \\
&\leq 2^{\frac{n'+1}{2}} b_H^{n'+1} \left\| \|h(\cdot, r_{n'+1})\|_{L^{\frac{1}{H}}(\Lambda_{r_{n'+1}}^{(n')})} \mathbb{1}_{]s,t[} \right\|_{L^{\frac{1}{H}}([0,T])} \\
&\leq 2^{\frac{n'+1}{2}} b_H^{n'+1} \|h\|_{\mathbb{1}_{]s,t[}} \left\| \mathbb{1}_{]s,t[} \right\|_{L^{\frac{1}{H}}(\Lambda_t^{(n'+1)})}.
\end{aligned}$$

This proves (12) for $n' + 1$.

The upper bound (13) for $n' + 1$ follows by applying Hölder's inequality, as we did for $n = 1$ in the first step of the proof. \square

Theorem 3.2

(A) *With the same hypothesis as in (a) of Proposition 3.1, the integral stochastic process $\mathbf{I}^{(n),H}(h) = \left\{ \mathbf{I}_t^{(n),H}(h); t \in [0, T] \right\}$, $n \geq 1$, is well defined as an iterated integral and for any $0 \leq s < t \leq T$,*

$$\mathbf{I}_t^{(n),H}(h) - \mathbf{I}_s^{(n),H}(h) = \int_{[0,t]^n} \left(K_{H,s,t}^{*,(n)} h \right) (\theta_1, \dots, \theta_n) dW_{\theta_1} \cdots dW_{\theta_n}, \quad (15)$$

with $K_{H,s,t}^{*,(n)}$, given in (10) and (11).

(B) Suppose the same hypotheses as in (b) of Proposition 3.1. Then, for any $p \in [2, \infty[$ and $0 \leq s < t \leq T$,

$$\left\| \mathbf{I}_t^{(n),H}(h) - \mathbf{I}_s^{(n),H}(h) \right\|_{L^p(\Omega)} \leq C \left\| K_{H,s,t}^{*,(n)} \right\|_{L^2([0,t]^n)} \leq C |t - s|^{H-1/q}, \quad (16)$$

for some positive constant C depending on q, p, h and H .

Consequently, the sample paths of $\mathbf{I}^{(n),H}(h)$ are γ -Hölder continuous with $\gamma \in]0, H - 1/q[$.

Proof: Let us show (A). We start by noticing that for $n = 1$ the equality (15) has already been met in Proposition 2.1 (see the proof of Proposition 3 in [1]).

Assume that (15) holds true up to an integer $n' \geq 1$. Let $h \in L^{\frac{1}{H}}(\Lambda_T^{(n'+1)})$. By the induction assumption, for any $r_{n'+1} \in [0, t]$, a.e., the random variable $\mathbf{I}_{r_{n'+1}}^{(n'),H}(h(\cdot, r_{n'+1}))$ is well defined as an iterated integral.

Fix $p \geq 2$. The rules of the Malliavin derivative for the standard Brownian motion and the hypercontractivity inequality (see for instance [9]) imply

$$\begin{aligned} \left\| \mathbf{I}_{r_{n'+1}}^{(n'),H}(h(\cdot, r_{n'+1})) \right\|_{1,p} &\leq C(n', p) \left\| K_{H,r_{n'+1}}^{*,(n')} h(\cdot, r_{n'+1}) \right\|_{L^2([0,r_{n'+1}]^{n'})} \\ &\leq C(n', p, H) \|h(\cdot, r_{n'+1})\|_{L^{\frac{1}{H}}(\Lambda_{r_{n'+1}}^{(n')})}, \end{aligned}$$

for any $r_{n'+1} \in [0, t]$, a.e.

Thus,

$$\begin{aligned} &\left(\int_0^T \left\| \mathbf{I}_{r_{n'+1}}^{(n'),H}(h(\cdot, r_{n'+1})) \right\|_{1,p}^{\frac{1}{H}} dr_{n'+1} \right)^H \\ &\leq C(n', p, H) \left(\int_0^T \|h(\cdot, r_{n'+1})\|_{L^{\frac{1}{H}}(\Lambda_{r_{n'+1}}^{(n')})}^{\frac{1}{H}} dr_{n'+1} \right)^H \\ &= C(n', p, H) \|h\|_{L^{\frac{1}{H}}(\Lambda_T^{(n'+1)})} < \infty. \end{aligned}$$

Thus, the process $\{\mathbf{I}_{r_{n'+1}}^{(n'),H}(h(\cdot, r_{n'+1})); r_{n'+1} \in [0, T]\}$ belongs to $L^{\frac{1}{H}}([0, T]; \mathbb{D}^{1,p})$, for any $p \geq 2$.

By applying Proposition 2.1, we can write

$$\mathbf{I}_t^{(n'+1),H}(h) - \mathbf{I}_s^{(n'+1),H}(h) = \int_0^T \left(K_{H,s,t}^{*,(1)} \mathbf{I}_{\star}^{(n'),H}(h(\cdot, \star)) \right) (\theta_{n'+1}) dW_{\theta_{n'+1}}.$$

Owing to (10), the last term can be decomposed into the sum of two terms denoted by M_1, M_2 , and defined as follows:

$$\begin{aligned} M_1 &= \int_s^t \left(\int_{\theta_{n'+1}}^t \mathbf{I}_{r_{n'+1}}^{(n'),H}(h(\cdot, r_{n'+1})) K_H(dr_{n'+1}, \theta_{n'+1}) \right) dW_{\theta_{n'+1}}, \\ M_2 &= \int_0^s \left(\int_s^t \mathbf{I}_{r_{n'+1}}^{(n'),H}(h(\cdot, r_{n'+1})) K_H(dr_{n'+1}, \theta_{n'+1}) \mathbb{1}_{\{s \neq 0\}} \right) dW_{\theta_{n'+1}}. \end{aligned}$$

We now apply the induction assumption to write all these terms by means of $(n'+1)$ -multiple integrals, obtaining

$$M_1 = \int_s^t \left(\int_{\theta_{n'+1}}^t \left(\int_{[0, r_{n'+1}]}^{n'} \left(K_{H, r_{n'+1}}^{*, (n')} h(\cdot, r_{n'+1}) \right) (\theta_1, \dots, \theta_{n'}) dW_{\theta_1} \cdots dW_{\theta_{n'}} \right) \times K_H(dr_{n'+1}, \theta_{n'+1}) \right) dW_{\theta_{n'+1}}, \quad (17)$$

$$M_2 = \int_0^s \left(\int_s^t \left(\int_{[0, r_{n'+1}]}^{n'} \left(K_{H, r_{n'+1}}^{*, (n')} h(\cdot, r_{n'+1}) \right) (\theta_1, \dots, \theta_{n'}) dW_{\theta_1} \cdots dW_{\theta_{n'}} \right) \times K_H(dr_{n'+1}, \theta_{n'+1}) \mathbb{1}_{\{s \neq 0\}} \right) dW_{\theta_{n'+1}}. \quad (18)$$

Finally, by the stochastic Fubini theorem applied to (17) and (18), we obtain

$$\begin{aligned} & \mathbf{I}_t^{(n'+1), H}(h) - \mathbf{I}_s^{(n'+1), H}(h) \\ &= \int_{[0, t]^{n'} \times]s, t[} \left(\int_{\bigvee_{i=1}^{n'+1} \theta_i}^t \left(K_{H, r_{n'+1}}^{*, (n')} h(\cdot, r_{n'+1}) \right) (\theta_1, \dots, \theta_{n'}) \right. \\ & \quad \times \left. K_H(dr_{n'+1}, \theta_{n'+1}) \right) dW_{\theta_1} \cdots dW_{\theta_{n'}} dW_{\theta_{n'+1}} \\ &+ \int_{[0, t]^{n'} \times]0, s[} \left(\int_{\bigvee_{i=1}^{n'+1} \theta_i \vee s}^t \left(K_{H, r_{n'+1}}^{*, (n')} h(\cdot, r_{n'+1}) \right) (\theta_1, \dots, \theta_{n'}) \right. \\ & \quad \times \left. K_H(dr_{n'+1}, \theta_{n'+1}) \mathbb{1}_{\{s \neq 0\}} \right) dW_{\theta_1} \cdots dW_{\theta_{n'}} dW_{\theta_{n'+1}}. \end{aligned}$$

That is,

$$\mathbf{I}_t^{(n'+1), H}(h) - \mathbf{I}_s^{(n'+1), H}(h) = \int_{[0, t]^{n'+1}} \left(K_{H, s, t}^{*, (n'+1)} h \right) (\theta_1, \dots, \theta_{n'+1}) dW_{\theta_1} \cdots dW_{\theta_{n'+1}}$$

(see (11)). This ends the proof of (15) and also that of (A).

For the proof of (B), we consider (15) and we apply the hypercontractivity inequality and (13). Then the conclusion on sample path regularity is a consequence of the Kolmogorov's criterion. \square

In the next section, we shall consider indefinite multiple integrals with integrands depending on the upper bound of the integration domain for which we shall apply the following Corollary.

Corollary 3.3 *Let $H \in]\frac{1}{2}, 1[$, $\beta \in]0, 1[$ and h be a measurable function defined on $[0, T]^{n+1}$ such that the mapping $t \mapsto h(\cdot, t)$ belongs to $\mathcal{C}^\beta([0, T]; L^q(\Lambda_T^{(n)}))$, for $qH > 1$. Then, the integral process $\{\mathbf{I}_t^{(n), H}(h(\cdot, t)), t \in [0, T]\}$ given in Theorem 3.2 has γ -Hölder continuous sample paths, a.s., with $\gamma \in]0, \beta \wedge (H - 1/q)[$.*

Proof: Let $p \in [2, \infty[$. The hypercontractivity inequality and (13) yield

$$\begin{aligned}
& \left\| \mathbf{I}_t^{(n),H} (h(\cdot, t)) - \mathbf{I}_s^{(n),H} (h(\cdot, s)) \right\|_{L^p(\Omega)} \\
& \leq \left\| \int_{[0,t]^n} \left(K_{H,s,t}^{*(n)} h(\cdot, t) \right) (\theta_1, \dots, \theta_n) dW_{\theta_1} \cdots dW_{\theta_n} \right\|_{L^p(\Omega)} \\
& \quad + \left\| \int_{[0,s]^n} \left(K_{H,s}^{*(n)} \left[\Delta^{n+1} h(\cdot, s; t) \right] \right) (\theta_1, \dots, \theta_n) dW_{\theta_1} \cdots dW_{\theta_n} \right\|_{L^p(\Omega)} \\
& \leq C(n, p) \left[\left\| K_{H,s,t}^{*(n)} h(\cdot, t) \right\|_{L^2([0,t]^n)} + \left\| K_{H,s}^{*(n)} \left[\Delta^{n+1} h(\cdot, s; t) \right] \right\|_{L^2([0,s]^n)} \right] \\
& \leq C(n, p, T, H) \left[\|h\|_{C^\beta([0,T]; L^q(\Lambda_T^{(n)}))} |t-s|^{H-1/q} + \|h\|_{C^\beta([0,T]; L^q(\Lambda_T^{(n)}))} |t-s|^\beta \right] \\
& \leq C(n, p, T, H) \|h\|_{C^\beta([0,T]; L^q(\Lambda_T^{(n)}))} |t-s|^{(H-1/q) \wedge \beta}.
\end{aligned}$$

The conclusion follows by applying Kolmogorov's criterion. \square

• **The case** $H \in]\frac{1}{4}, \frac{1}{2}[$.

Let us introduce the functions that will appear as kernels of increments of the indefinite multiple integrals. Fix $0 \leq s < t \leq T$. We set

$$\begin{aligned}
\left(K_{H,s,t}^{*(1)} h \right) (\theta_1) &= h(\theta_1) K_H(t, \theta_1) \mathbb{1}_{]s,t[}(\theta_1) \\
& \quad + \int_{\theta_1}^t [h(r_1) - h(\theta_1)] K_H(dr_1, \theta_1) \mathbb{1}_{]s,t[}(\theta_1) \\
& \quad + \int_s^t h(r_1) K_H(dr_1, \theta_1) \mathbb{1}_{]0,s[}(\theta_1) \mathbb{1}_{\{s \neq 0\}},
\end{aligned} \tag{19}$$

and for any integer $n \geq 2$,

$$\begin{aligned}
& \left(K_{H,s,t}^{*(n)} h \right) (\theta_1, \dots, \theta_n) \\
&= \left(K_{H,\theta_n}^{*(n-1)} h(\cdot, \theta_n) \right) (\theta_1, \dots, \theta_{n-1}) K_H(t, \theta_n) \mathbb{1}_{[0,\theta_n]^{n-1}}(\theta_1, \dots, \theta_{n-1}) \mathbb{1}_{]s,t[}(\theta_n) \\
& \quad + \int_{\theta_n}^t \left(K_{H,\theta_n}^{*(n-1)} [h(\cdot, r_n) - h(\cdot, \theta_n)] \right) (\theta_1, \dots, \theta_{n-1}) K_H(dr_n, \theta_n) \\
& \quad \quad \times \mathbb{1}_{[0,\theta_n]^{n-1}}(\theta_1, \dots, \theta_{n-1}) \mathbb{1}_{]s,t[}(\theta_n) \\
& \quad + \int_{\bigvee_{i=1}^n \theta_i}^t \left(K_{H,\theta_n, r_n}^{*(n-1)} h(\cdot, r_n) \right) (\theta_1, \dots, \theta_{n-1}) K_H(dr_n, \theta_n) \\
& \quad \quad \times \mathbb{1}_{[0,t]^{n-1}}(\theta_1, \dots, \theta_{n-1}) \mathbb{1}_{]s,t[}(\theta_n) \\
& \quad + \int_{\bigvee_{i=1}^{n-1} \theta_i \vee s}^t \left(K_{H,r_n}^{*(n-1)} h(\cdot, r_n) \right) (\theta_1, \dots, \theta_{n-1}) K_H(dr_n, \theta_n) \\
& \quad \quad \times \mathbb{1}_{[0,t]^{n-1}}(\theta_1, \dots, \theta_{n-1}) \mathbb{1}_{]0,s[}(\theta_n) \mathbb{1}_{\{s \neq 0\}},
\end{aligned} \tag{20}$$

where we write $K_{H,\cdot}^{*(m)}$ instead of $K_{H,0,\cdot}^{*(m)}$.

Proposition 3.4 Fix $H \in]\frac{1}{4}, \frac{1}{2}[$ and a natural number $n \geq 1$. Let $h \in \mathbb{H}^\lambda(\Lambda_T^{(n)})$, for some λ satisfying $\lambda + H > \frac{1}{2}$. Then, for $0 \leq s < t \leq T$,

$$\left\| K_{H,s,t}^{*,(n)} h \right\|_{L^2([0,t]^n)} \leq C(T, \lambda, H) \|h\|_{\mathbb{H}^\lambda(\Lambda_t^{(n)})} |t - s|^H, \quad (21)$$

with some positive constant $C(T, \lambda, H)$. Thus, $K_{H,s,t}^{*,(n)} : \mathbb{H}^\lambda(\Lambda_t^{(n)}) \longrightarrow L^2([0,t]^n)$ defined in (19) and (20) is a linear continuous operator.

Proof: Let $n = 1$. Owing to (19),

$$\begin{aligned} & \left\| K_{H,s,t}^{*,(1)} h \right\|_{L^2([0,t])} \\ & \leq \left\| h(\cdot) K_H(t, \cdot) \mathbb{1}_{]s,t[} \right\|_{L^2([0,t])} + \left\| \int_{\cdot}^t [\Delta^1 h(\cdot; r_1)] K_H(dr_1, \cdot) \mathbb{1}_{]s,t[} \right\|_{L^2([0,t])} \\ & \quad + \left\| \int_s^t h(r_1) K_H(dr_1, \cdot) \mathbb{1}_{]0,s[} \mathbb{1}_{\{s \neq 0\}} \right\|_{L^2([0,t])} \\ & \leq \|h\|_{\mathbb{H}^\lambda(\Lambda_t^{(1)})} \left[\left\| K_H(t, \cdot) \mathbb{1}_{]s,t[} \right\|_{L^2([0,t])} + \left\| \int_{\cdot}^t |r_1 - \cdot|^\lambda |K_H(dr_1, \cdot)| \mathbb{1}_{]s,t[} \right\|_{L^2([0,t])} \right. \\ & \quad \left. + \left\| \int_s^t |K_H(dr_1, \cdot)| \mathbb{1}_{]0,s[} \mathbb{1}_{\{s \neq 0\}} \right\|_{L^2([0,t])} \right] \\ & \leq C(T, \lambda, H) \|h\|_{\mathbb{H}^\lambda(\Lambda_t^{(1)})} |t - s|^H, \end{aligned}$$

where in the last estimate we have used (4). Thus, (21) holds for $n = 1$.

Suppose now that (21) holds up to an integer $n' \geq 1$. Let $h \in \mathbb{H}^\lambda(\Lambda_t^{(n'+1)})$. The functions $h(\cdot, \eta_{n'+1})$ and $\Delta^{n'+1} h(\cdot, \eta_{n'+1}; \tau_{n'+1})$, for any fixed $\eta_{n'+1} \leq t$ and $\tau_{n'+1} < \eta_{n'+1} \leq t$, respectively, belong to $\mathbb{H}^\lambda(\Lambda_{\eta_{n'+1}}^{(n')})$. From (20) it follows that $\left\| K_{H,s,t}^{*,(n'+1)} h \right\|_{L^2([0,t]^{n'+1})}^2 \leq 2^3 \left(\sum_{i=1}^4 R_i \right)$, with

$$\begin{aligned} R_1 &= \int_{[0, \theta_{n'+1}]^{n'} \times]s,t[} \left(K_{H, \theta_{n'+1}}^{*,(n')} h(\cdot, \theta_{n'+1}) \right)^2 (\theta_1, \dots, \theta_{n'}) \\ & \quad \times |K_H(t, \theta_{n'+1})|^2 d\theta_1 \cdots d\theta_{n'} d\theta_{n'+1}; \\ R_2 &= \int_{[0, \theta_{n'+1}]^{n'} \times]s,t[} \left(\int_{\theta_{n'+1}}^t \left(K_{H, \theta_{n'+1}}^{*,(n')} [\Delta^{n'+1} h(\cdot, \theta_{n'+1}; r_{n'+1})] \right) \right) (\theta_1, \dots, \theta_{n'}) \\ & \quad \times |K_H(dr_{n'+1}, \theta_{n'+1})|^2 d\theta_1 \cdots d\theta_{n'} d\theta_{n'+1}; \\ R_3 &= \int_{[0,t]^{n'} \times]s,t[} \left(\int_{\bigvee_{i=1}^{n'} \theta_i}^t \left(K_{H, \theta_{n'+1}, r_{n'+1}}^{*,(n')} h(\cdot, r_{n'+1}) \right) \right) (\theta_1, \dots, \theta_{n'}) \\ & \quad \times |K_H(dr_{n'+1}, \theta_{n'+1})|^2 d\theta_1 \cdots d\theta_{n'} d\theta_{n'+1}; \\ R_4 &= \int_{[0,t]^{n'} \times]0,s[} \left(\int_{\bigvee_{i=1}^{n'} \theta_i \vee s}^t \left(K_{H, r_{n'+1}}^{*,(n')} h(\cdot, r_{n'+1}) \right) \right) (\theta_1, \dots, \theta_{n'}) \\ & \quad \times |K_H(dr_{n'+1}, \theta_{n'+1})|^2 d\theta_1 \cdots d\theta_{n'} d\theta_{n'+1}. \end{aligned}$$

Applying Minkowski's inequality to each one of these terms and the induction assumption, we obtain,

$$\begin{aligned}
R_1 &\leq \int_s^t \left\| K_{H, \theta_{n'+1}}^{*, (n')} h(\cdot, \theta_{n'+1}) \right\|_{L^2([0, \theta_{n'+1}]^{n'})}^2 K_H(t, \theta_{n'+1})^2 d\theta_{n'+1} \\
&\leq C(T, \lambda, H) \int_s^t \|h(\cdot, \theta_{n'+1})\|_{\mathbb{H}^\lambda(\Lambda_{\theta_{n'+1}}^{(n')})}^2 K_H(t, \theta_{n'+1})^2 d\theta_{n'+1} \\
&\leq C(T, \lambda, H) \|h\|_{\mathbb{H}^\lambda(\Lambda_t^{(n'+1)})}^2 |t - s|^{2H}, \tag{22}
\end{aligned}$$

$$\begin{aligned}
R_2 &\leq \int_s^t \left(\int_{\theta_{n'+1}}^t \left\| K_{H, \theta_{n'+1}}^{*, (n')} [\Delta^{n'+1} h(\cdot, \theta_{n'+1}; r_{n'+1})] \right\|_{L^2([0, \theta_{n'+1}]^{n'})} \right. \\
&\quad \left. \times |K_H(dr_{n'+1}, \theta_{n'+1})| \right)^2 d\theta_{n'+1} \\
&\leq C(T, \lambda, H) \int_s^t \left(\int_{\theta_{n'+1}}^t \left\| \Delta^{n'+1} h(\cdot, \theta_{n'+1}; r_{n'+1}) \right\|_{\mathbb{H}^\lambda(\Lambda_{\theta_{n'+1}}^{(n')})} \right. \\
&\quad \left. \times |K_H(dr_{n'+1}, \theta_{n'+1})| \right)^2 d\theta_{n'+1} \\
&\leq C(T, \lambda, H) \|h\|_{\mathbb{H}^\lambda(\Lambda_t^{(n'+1)})}^2 \\
&\quad \times \int_s^t \left(\int_{\theta_{n'+1}}^t |r_{n'+1} - \theta_{n'+1}|^\lambda |K_H(dr_{n'+1}, \theta_{n'+1})| \right)^2 d\theta_{n'+1} \\
&\leq C(T, \lambda, H) \|h\|_{\mathbb{H}^\lambda(\Lambda_t^{(n'+1)})}^2 |t - s|^{2(H+\lambda)}, \tag{23}
\end{aligned}$$

$$\begin{aligned}
R_3 &\leq \int_s^t \left(\int_{\theta_{n'+1}}^t \left\| K_{H, \theta_{n'+1}, r_{n'+1}}^{*, (n')} h(\cdot, r_{n'+1}) \right\|_{L^2([0, r_{n'+1}]^{n'})} \right. \\
&\quad \left. \times |K_H(dr_{n'+1}, \theta_{n'+1})| \right)^2 d\theta_{n'+1} \\
&\leq C(T, \lambda, H) \int_s^t \left(\int_{\theta_{n'+1}}^t \|h(\cdot, r_{n'+1})\|_{\mathbb{H}^\lambda(\Lambda_{r_{n'+1}}^{(n')})} \right. \\
&\quad \left. \times |r_{n'+1} - \theta_{n'+1}|^H |K_H(dr_{n'+1}, \theta_{n'+1})| \right)^2 d\theta_{n'+1} \\
&\leq C(T, \lambda, H) \|h\|_{\mathbb{H}^\lambda(\Lambda_t^{(n'+1)})}^2 \\
&\quad \times \int_s^t \left(\int_{\theta_{n'+1}}^t |r_{n'+1} - \theta_{n'+1}|^H |K_H(dr_{n'+1}, \theta_{n'+1})| \right)^2 d\theta_{n'+1} \\
&\leq C(T, \lambda, H) \|h\|_{\mathbb{H}^\lambda(\Lambda_t^{(n'+1)})}^2 |t - s|^{4H}, \tag{24}
\end{aligned}$$

$$\begin{aligned}
R_4 &\leq \int_0^s \left(\int_s^t \left\| K_{H,r_{n'+1}}^{*,(n')} h(\cdot, r_{n'+1}) \right\|_{L^2([0,r_{n'+1}]^{n'})} \right. \\
&\quad \left. \times |K_H(dr_{n'+1}, \theta_{n'+1})| \right)^2 d\theta_{n'+1} \\
&\leq C(T, \lambda, H) \int_0^s \left(\int_s^t \|h(\cdot, r_{n'+1})\|_{\mathbb{H}^\lambda(\Lambda_{r_{n'+1}}^{(n')})} |K_H(dr_{n'+1}, \theta_{n'+1})| \right)^2 d\theta_{n'+1} \\
&\leq C(T, \lambda, H) \|h\|_{\mathbb{H}^\lambda(\Lambda_t^{(n'+1)})}^2 \int_0^s \left(\int_s^t |K_H(dr_{n'+1}, \theta_{n'+1})| \right)^2 d\theta_{n'+1} \\
&\leq C(T, \lambda, H) \|h\|_{\mathbb{H}^\lambda(\Lambda_t^{(n'+1)})}^2 |t-s|^{2H}. \tag{25}
\end{aligned}$$

With (22), (23), (24), (25), we see that (21) holds for $n = n' + 1$ and this ends the proof of this proposition. \square

Theorem 3.5 *With the same hypotheses as in Proposition 3.4, the indefinite integral stochastic process $\mathbf{I}^{(n),H}(h) = \{\mathbf{I}_t^{(n),H}(h), t \in [0, T]\}$ is well defined as an iterated integral. Moreover, for any $0 \leq s < t \leq T$,*

$$\mathbf{I}_t^{(n),H}(h) - \mathbf{I}_s^{(n),H}(h) = \int_{[0,t]^{n'}} \left(K_{H,s,t}^{*,(n)} h \right) (\theta_1, \dots, \theta_n) dW_{\theta_1} \cdots dW_{\theta_n}, \tag{26}$$

with $K_{H,s,t}^{*,(n)}$, given in (19) and (20).

Thus, for any $p \in [2, \infty[$,

$$\left\| \mathbf{I}_t^{(n),H}(h) - \mathbf{I}_s^{(n),H}(h) \right\|_{L^p(\Omega)} \leq C \left\| K_{H,s,t}^{*,(n)} h \right\|_{L^2([0,t]^{n'})} \leq C |t-s|^H, \tag{27}$$

for some positive constant C depending on p, h, T, λ and H .

Consequently, the sample paths of $\mathbf{I}^{(n),H}(h)$ are γ -Hölder continuous with $\gamma \in]0, H[$.

Proof: Let us prove first (26). For $n = 1$, it is an immediate consequence of Proposition 2.2. The formula (26) is given in the proof of Proposition 1 of [1].

Assume that (26) holds up to an integer $n' \geq 1$. Consider a function $h \in \mathbb{H}^\lambda(\Lambda_T^{(n'+1)})$. For any $0 \leq s < t \leq T$, we can write

$$\begin{aligned}
&\mathbf{I}_t^{(n'),H}(h(\cdot, t)) - \mathbf{I}_s^{(n'),H}(h(\cdot, s)) \\
&= \mathbf{I}_t^{(n'),H}(h(\cdot, t)) - \mathbf{I}_s^{(n'),H}(h(\cdot, t)) + \mathbf{I}_s^{(n'),H} \left(\left[\Delta^{n'+1} h(\cdot, s; t) \right] \right). \tag{28}
\end{aligned}$$

By the induction assumption, the following representations hold:

$$\begin{aligned}
&\mathbf{I}_t^{(n'),H}(h(\cdot, t)) - \mathbf{I}_s^{(n'),H}(h(\cdot, t)) \\
&= \int_{[0,t]^{n'}} \left(K_{H,s,t}^{*,(n')} h(\cdot, t) \right) (\theta_1, \dots, \theta_{n'}) dW_{\theta_1} \cdots dW_{\theta_{n'}}, \\
&\mathbf{I}_s^{(n'),H} \left(\left[\Delta^{n'+1} h(\cdot, s; t) \right] \right) \\
&= \int_{[0,s]^{n'}} \left(K_{H,s}^{*,(n')} \left[\Delta^{n'+1} h(\cdot, s; t) \right] \right) (\theta_1, \dots, \theta_{n'}) dW_{\theta_1} \cdots dW_{\theta_{n'}}.
\end{aligned}$$

We are going to prove that the above integrands satisfy the hypotheses of Proposition 2.2. Indeed, for any $p \in [2, \infty]$, the rules of the Malliavin derivative, the hypercontractivity inequality and (21) yield the following.

$$\begin{aligned}
& \left\| \mathbf{I}_t^{(n'),H} (h(\cdot, t)) - \mathbf{I}_s^{(n'),H} (h(\cdot, s)) \right\|_{\mathbb{D}^{1,p}} \\
& \leq \left\| \int_{[0,t]^{n'}} \left(K_{H,s,t}^{*,(n')} h(\cdot, t) \right) (\theta_1, \dots, \theta_{n'}) dW_{\theta_1} \cdots dW_{\theta_{n'}} \right\|_{\mathbb{D}^{1,p}} \\
& \quad + \left\| \int_{[0,s]^{n'}} \left(K_{H,s}^{*,(n')} [\Delta^{n'+1} h(\cdot, s; t)] \right) (\theta_1, \dots, \theta_{n'}) dW_{\theta_1} \cdots dW_{\theta_{n'}} \right\|_{\mathbb{D}^{1,p}} \\
& \leq C(n', p) \left[\left\| K_{H,s,t}^{*,(n')} h(\cdot, t) \right\|_{L^2([0,t]^{n'})} + \left\| K_{H,s}^{*,(n')} [\Delta^{n'+1} h(\cdot, s; t)] \right\|_{L^2([0,s]^{n'})} \right] \\
& \leq C(n', p, T, \lambda, H) \left[\|h\|_{\mathbb{H}^\lambda(\Lambda_t^{(n')})} |t-s|^H + \|h\|_{\mathbb{H}^\lambda(\Lambda_t^{(n')})} |t-s|^\lambda \right] \\
& \leq C(n', p, T, \lambda, H) \|h\|_{\mathbb{H}^\lambda(\Lambda_t^{(n')})} |t-s|^{H \wedge \lambda}.
\end{aligned} \tag{29}$$

Thus, the $\mathbb{D}^{1,p}$ -valued stochastic process $Y^{(n')} = \left\{ \mathbf{I}_t^{(n'),H} (h(\cdot, t)); t \in [0, T] \right\}$ has $(H \wedge \lambda)$ -Hölder continuous sample paths, a.s. Notice that $(H \wedge \lambda) + H > \frac{1}{2}$. By applying Proposition 2.2, we can write

$$\begin{aligned}
\mathbf{I}_t^{(n'+1),H} (h) - \mathbf{I}_s^{(n'+1),H} (h) &= \int_0^t \left(K_{H,t}^{*,(1)} \mathbf{I}_\star^{(n'),H} (h(\cdot, \star)) \right) (\theta_{n'+1}) dW_{\theta_{n'+1}} \\
&\quad - \int_0^s \left(K_{H,s}^{*,(1)} \mathbf{I}_\star^{(n'),H} (h(\cdot, \star)) \right) (\theta_{n'+1}) dW_{\theta_{n'+1}}.
\end{aligned}$$

Owing to (19) applied to $h(\theta) := \mathbf{I}_\theta^{(n'),H} (h(\cdot, \theta))$ and the identity (28), the last term can be decomposed into the sum of four terms denoted by N_i , $i = 1, \dots, 4$, and defined as follows:

$$\begin{aligned}
N_1 &= \int_s^t \mathbf{I}_{\theta_{n'+1}}^{(n'),H} (h(\cdot, \theta_{n'+1})) K_H(t, \theta_{n'+1}) dW_{\theta_{n'+1}}, \\
N_2 &= \int_s^t \left(\int_{\theta_{n'+1}}^t \left[\mathbf{I}_{r_{n'+1}}^{(n'),H} (h(\cdot, r_{n'+1})) - \mathbf{I}_{\theta_{n'+1}}^{(n'),H} (h(\cdot, r_{n'+1})) \right] K_H(dr_{n'+1}, \theta_{n'+1}) \right) dW_{\theta_{n'+1}}, \\
N_3 &= \int_s^t \left(\int_{\theta_{n'+1}}^t \mathbf{I}_{\theta_{n'+1}}^{(n'),H} (\Delta^{n'+1} h(\cdot, \theta_{n'+1}; r_{n'+1})) K_H(dr_{n'+1}, \theta_{n'+1}) \right) dW_{\theta_{n'+1}}, \\
N_4 &= \int_0^s \left(\int_s^t \mathbf{I}_{r_{n'+1}}^{(n'),H} (h(\cdot, r_{n'+1})) K_H(dr_{n'+1}, \theta_{n'+1}) \mathbb{1}_{\{s \neq 0\}} \right) dW_{\theta_{n'+1}}.
\end{aligned}$$

We now apply the induction assumption to write all these terms by means of $(n'+1)$ -

multiple integrals, obtaining

$$\begin{aligned}
N_1 &= \int_s^t \left(\int_{[0, \theta_{n'+1}]^{n'}} \left(K_{H, \theta_{n'+1}}^{*, (n')} h(\cdot, \theta_{n'+1}) \right) (\theta_1, \dots, \theta_{n'}) dW_{\theta_1} \cdots dW_{\theta_{n'}} \right) \\
&\quad \times K_H(t, \theta_{n'+1}) dW_{\theta_{n'+1}}, \\
N_2 &= \int_s^t \left(\int_{\theta_{n'+1}}^t \left(\int_{[0, r_{n'+1}]^{n'}} \left(K_{H, \theta_{n'+1}, r_{n'+1}}^{*, (n')} h(\cdot, r_{n'+1}) \right) (\theta_1, \dots, \theta_{n'}) dW_{\theta_1} \cdots dW_{\theta_{n'}} \right) \right. \\
&\quad \left. \times K_H(dr_{n'+1}, \theta_{n'+1}) \right) dW_{\theta_{n'+1}}, \tag{30}
\end{aligned}$$

$$\begin{aligned}
N_3 &= \int_s^t \left(\int_{\theta_{n'+1}}^t \left(\int_{[0, \theta_{n'+1}]^{n'}} \left(K_{H, \theta_{n'+1}}^{*, (n')} [\Delta^{n'+1} h(\cdot, \theta_{n'+1}; r_{n'+1})] \right) (\theta_1, \dots, \theta_{n'}) \right. \right. \\
&\quad \left. \left. \times dW_{\theta_1} \cdots dW_{\theta_{n'}} \right) K_H(dr_{n'+1}, \theta_{n'+1}) \right) dW_{\theta_{n'+1}}, \tag{31}
\end{aligned}$$

$$\begin{aligned}
N_4 &= \int_0^s \left(\int_s^t \left(\int_{[0, r_{n'+1}]^{n'}} \left(K_{H, r_{n'+1}}^{*, (n')} h(\cdot, r_{n'+1}) \right) (\theta_1, \dots, \theta_{n'}) dW_{\theta_1} \cdots dW_{\theta_{n'}} \right) \right. \\
&\quad \left. \times K_H(dr_{n'+1}, \theta_{n'+1}) \mathbb{1}_{\{s \neq 0\}} \right) dW_{\theta_{n'+1}}. \tag{32}
\end{aligned}$$

Finally, by the stochastic Fubini theorem applied to (30), (32) and (31), we obtain

$$\begin{aligned}
&\mathbf{I}_t^{(n'+1), H}(h) - \mathbf{I}_s^{(n'+1), H}(h) \\
&= \int_{[0, \theta_{n'+1}]^{n'} \times]s, t[} \left(K_{H, \theta_{n'+1}}^{*, (n')} h(\cdot, \theta_{n'+1}) \right) (\theta_1, \dots, \theta_{n'}) K_H(t, \theta_{n'+1}) \\
&\quad \times dW_{\theta_1} \cdots dW_{\theta_{n'}} dW_{\theta_{n'+1}} \\
&\quad + \int_{[0, t]^{n'} \times]s, t[} \left(\int_{\bigvee_{i=1}^{n'} \theta_i}^t \left(K_{H, \theta_{n'+1}, r_{n'+1}}^{*, (n')} h(\cdot, r_{n'+1}) \right) (\theta_1, \dots, \theta_{n'}) \right. \\
&\quad \left. \times K_H(dr_{n'+1}, \theta_{n'+1}) \right) dW_{\theta_1} \cdots dW_{\theta_{n'}} dW_{\theta_{n'+1}} \\
&\quad + \int_{[0, \theta_{n'+1}]^{n'} \times]s, t[} \left(\int_{\theta_{n'+1}}^t \left(K_{H, \theta_{n'+1}}^{*, (n')} [\Delta^{n'+1} h(\cdot, \theta_{n'+1}; r_{n'+1})] \right) (\theta_1, \dots, \theta_{n'}) \right. \\
&\quad \left. \times K_H(dr_{n'+1}, \theta_{n'+1}) \right) dW_{\theta_1} \cdots dW_{\theta_{n'}} dW_{\theta_{n'+1}} \\
&\quad + \int_{[0, t]^{n'} \times]0, s[} \left(\int_{\bigvee_{i=1}^{n'} \theta_i \vee s}^t \left(K_{H, r_{n'+1}}^{*, (n')} h(\cdot, r_{n'+1}) \right) (\theta_1, \dots, \theta_{n'}) \right. \\
&\quad \left. \times K_H(dr_{n'+1}, \theta_{n'+1}) \mathbb{1}_{\{s \neq 0\}} \right) dW_{\theta_1} \cdots dW_{\theta_{n'}} dW_{\theta_{n'+1}}.
\end{aligned}$$

That is,

$$\mathbf{I}_t^{(n'+1), H}(h) - \mathbf{I}_s^{(n'+1), H}(h) = \int_{[0, t]^{n'+1}} \left(K_{H, s, t}^{*, (n'+1)} h \right) (\theta_1, \dots, \theta_{n'+1}) dW_{\theta_1} \cdots dW_{\theta_{n'+1}}$$

(see (20)). This ends the proof of (26).

The upper bound estimate (27) follows from (26), the hypercontractivity inequality and (21), while the conclusion on sample path regularity is a consequence of the Kolmogorov's criterion. \square

This is the analogue of Corollary 3.3 for $H \in]\frac{1}{4}, \frac{1}{2}[$.

Corollary 3.6 *Let $H \in]\frac{1}{4}, \frac{1}{2}[$, $\lambda \in]0, 1[$ be such that $\lambda + H > \frac{1}{2}$ and $\beta \in]0, 1[$. Let h be a measurable function defined on $[0, T]^{n+1}$ such that the mapping $t \mapsto h(\cdot, t)$ belongs to $\mathcal{C}^\beta([0, T]; \mathbb{H}^\lambda(\Lambda_T^{(n)}))$. Then, the integral process $\{\mathbf{I}_t^{(n), H}(h(\cdot, t)), t \in [0, T]\}$ given in Theorem 3.5 has γ -Hölder continuous sample paths, a.s., with $\gamma \in]0, \beta \wedge H[$.*

Proof: It follows from the estimate (29), with λ replaced now by β , and Kolmogorov's criterion. \square

4 Large deviation principle for Banach valued multiple integrals

In this section we fix an integer $n \geq 1$ and we consider the integral processes $\{\mathbf{I}_t^{(n), H}(h(\cdot, t)), t \in [0, T]\}$ given in the Corollaries 3.3 and 3.6, respectively, for a fixed deterministic function h . The fractional Brownian motion B^H is replaced by $\sqrt{\varepsilon}B^H$, $\varepsilon \in]0, \infty[$. Since the integrand h is deterministic, we obtain a family $\{\varepsilon^{\frac{n}{2}}\mathbf{I}_t^{(n), H}(h(\cdot, t)), t \in [0, T]\}$ indexed by $\varepsilon \in]0, \infty[$. It can also be thought of as a family of random variables taking values on the space $\mathcal{C}^\gamma([0, T])$, for some value of $\gamma \in]0, 1[$ made explicit in the above mentioned corollaries. Throughout this section we shall denote by $\{\mathbf{I}_h^{(n), H, \varepsilon}, \varepsilon > 0\}$ this object. Our purpose is to establish a large deviation principle for the corresponding family of probability laws. For this, we shall apply the results of [7].

The first step consists of describing the abstract Wiener space to be considered in our problem. First, we should replace the space of Hölder continuous functions by a Polish space by a classical procedure (see for instance [3]). More precisely, fix $\gamma \in]0, 1[$ and for every $0 < \delta \leq T$ set

$$\omega_x(\delta) = \sup_{\substack{|t-s| \leq \delta \\ 0 \leq s, t \leq T \\ s \neq t}} \frac{|x(t) - x(s)|}{|t - s|^\gamma}.$$

Let $\mathcal{C}^{\gamma, 0}([0, T])$ be the subspace of $\mathcal{C}^\gamma([0, T])$ consisting of functions such that $\lim_{\delta \rightarrow 0^+} \omega_x(\delta) = 0$. The space $\mathcal{C}^{\gamma, 0}([0, T])$ is a closed subspace of $\mathcal{C}^\gamma([0, T])$, so that endowed with the norm $\|\cdot\|_\gamma$ it is a separable Banach space, whereas $\mathcal{C}^\gamma([0, T])$ is not.

It is easy to check that $\mathcal{C}^\gamma([0, T]) \subset \mathcal{C}^{\gamma', 0}([0, T])$, for any $\gamma' < \gamma$.

Consider the set

$$\mathcal{H}^H = \left\{ \varphi : [0, T] \rightarrow \mathbb{R} : \varphi(t) = \int_0^t K_H(t, \theta) \dot{\varphi}(\theta) d\theta, \dot{\varphi} \in L^2([0, T]) \right\}, \quad (33)$$

with K_H given in (2), endowed with the inner product $\langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}^H} = \langle \dot{\varphi}_1, \dot{\varphi}_2 \rangle_{L^2([0, T])}$. This is the reproducing kernel Hilbert space of the fractional Brownian motion, as it is proved in [6].

Remark 4.1 For any $H \in]\frac{1}{2}, 1[$, we have

$$L^2([0, T]) \subset L^{\frac{1}{H}}([0, T]) \subset |\mathcal{H}^H| \subset \mathcal{H}^H. \quad (34)$$

Indeed, the first inclusion is obvious, while the second one follows from the estimate (9), and the last one is pointed out in [17]. This implies that \mathcal{H}^H contains the set of continuous functions defined on $[0, T]$.

For $H \in]0, \frac{1}{2}[$, $\lambda \in]0, 1[$ such that $\lambda + H > \frac{1}{2}$, as is pointed out in [14],

$$\mathcal{C}^\lambda([0, T]) \subset \mathcal{H}^H. \quad (35)$$

As a consequence, \mathcal{H}^H contains the set of Lipschitz continuous functions.

Let $\mathcal{C}_0([0, T])$ be the set of continuous functions defined on $[0, T]$ vanishing at the origin, $\bar{i} : \mathcal{H}^H \hookrightarrow \mathcal{C}_0([0, T])$ the canonical inclusion, and \mathbf{P}^H be the law of the fractional Brownian motion on $\mathcal{C}_0([0, T])$. The triple $(\mathcal{C}_0([0, T]), \mathcal{H}^H, \bar{i}, \mathbf{P}^H)$ is an abstract Wiener space (see [6] for the proof of these results). This result can be strengthened, as is stated in the next Proposition.

Proposition 4.1 For any $\zeta < H$, the set \mathcal{H}^H is included in $\mathcal{C}^{\zeta, 0}([0, T])$. Moreover, denoting by $i : \mathcal{H}^H \hookrightarrow \mathcal{C}^{\zeta, 0}([0, T])$ the canonical embedding, the quadruple

$$(\mathcal{C}^{\zeta, 0}([0, T]), \mathcal{H}^H, i, \mathbf{P}^H),$$

is an abstract Wiener space.

Proof: Indeed, let $\varphi \in \mathcal{H}^H$ and $0 \leq s < t \leq T$. Schwarz's inequality yields

$$\begin{aligned} |\varphi(t) - \varphi(s)| &= \left| \int_0^t (K_H(t, r) - K_H(s, r)) \dot{\varphi}(r) dr \right| \\ &\leq \|\varphi\|_{\mathcal{H}^H} \|K_H(t, \cdot) - K_H(s, \cdot)\|_{L^2([0, T])} \leq \|\varphi\|_{\mathcal{H}^H} C |t - s|^H. \end{aligned}$$

Consequently,

$$\|\varphi\|_{\mathcal{C}^H([0, T])} \leq C \|\varphi\|_{\mathcal{H}^H}.$$

This proves that the mapping i is a continuous embedding.

Fix $x \in \mathcal{C}^{\zeta, 0}([0, T])$ and consider the approximating sequence, $\{x^{(n)}, n \in \mathbb{N}\}$, consisting of linear interpolations on the dyadic numbers. That is,

$$x^{(n)}(t) = \sum_{j=0}^{2^n-1} \left[x(t_j^n) + \frac{2^n}{T} (t - t_j^n) (x(t_{j+1}^n) - x(t_j^n)) \right] \mathbb{1}_{[t_j^n, t_{j+1}^n]}(t), \quad (36)$$

where $t_j^n = \frac{jT}{2^n}$, $j = 0, \dots, 2^n$. Clearly, each x^n is a Lipschitz function, consequently $\{x^{(n)}, n \in \mathbb{N}\} \subset \mathcal{H}^H$ (see Remark 4.1).

It is an easy exercise to check that this sequence converges to x in the norm $\|\cdot\|_\zeta$. Therefore, $i(\mathcal{H}^H)$ is dense in $\mathcal{C}^{\zeta, 0}([0, T])$.

Finally, since the trajectories of the fractional Brownian motion are a.s. in $\mathcal{C}^{\zeta, 0}([0, T])$, for any $\zeta \in]0, H[$, we have $(\mathbf{P}^H)^*(\mathcal{C}^{\zeta, 0}([0, T])) = 1$, where $(\mathbf{P}^H)^*$ denotes the exterior measure. This leads to the conclusion (see Theorem 2.4 of [2]). \square

Remark 4.2 *An alternate approximation sequence by Lipschitz functions for $x \in \mathcal{C}^{\zeta,0}([0, T])$ in the norm $\|\cdot\|_{\zeta}$, is provided by*

$$x^{(n)}(t) = \frac{n}{T} \int_t^{t+\frac{T}{n}} x(\tau) d\tau - \frac{n}{T} \int_0^{\frac{T}{n}} x(\tau) d\tau,$$

with $x(t) = x(T)$ for $t \geq T$ (see [13], page 276).

In the sequel, we consider as reference probability space the triple $(\mathcal{C}^{\gamma,0}([0, T]), \mathcal{B}, \mathbf{P}^H)$, where \mathcal{B} is the Borel σ -field of $\mathcal{C}^{\gamma,0}([0, T])$. In particular, the random variables $\{\mathbf{I}_h^{(n),H,\varepsilon}, \varepsilon > 0\}$ and $\mathbf{I}_h^{(n),H} := \mathbf{I}_h^{(n),H,1}$ are supposed to be defined in this probability space. Under the hypotheses of Corollaries 3.3 and 3.6, depending whether $H \in]\frac{1}{2}, 1[$ or $H \in]\frac{1}{4}, \frac{1}{2}[$, they are $\mathcal{C}^{\gamma,0}([0, T])$ -valued random variables with $\gamma \in]0, \beta_0[$, where $\beta_0 = \beta \wedge (H - 1/q)$ if $H \in]\frac{1}{2}, 1[$, $qH > 1$, and $\beta_0 = \beta \wedge H$, if $H \in]\frac{1}{4}, \frac{1}{2}[$, respectively. We also denote by $\mathbf{I}_h^{(n),H}(B^H + \varphi)$, for $\varphi \in \mathcal{H}^H$, the multiple integral with respect to $B^H + \varphi$ instead of B^H .

The main theorem of [7] (see page 4) applied to the abstract Wiener space given in Proposition 4.1, the separable Banach space $B = \mathcal{C}^{\gamma,0}([0, T])$ and the random variable $\mathbf{I}_h^{(n),H}$ in the $\mathcal{C}^{\gamma,0}([0, T])$ -valued Wiener chaos of degree n , implies the large deviation principle for the family of laws of $\{\mathbf{I}_h^{(n),H,\varepsilon}, \varepsilon > 0\}$ given here.

Theorem 4.2 *Each one of the two sets of assumptions*

- (i) $H \in]\frac{1}{2}, 1[$, $qH > 1$, $\beta \in]0, 1[$, h is a measurable function defined on $[0, T]^{n+1}$ such that the mapping $t \mapsto h(\cdot, t)$ belongs to $\mathcal{C}^{\beta}([0, T]; L^q(\Lambda_T^{(n)}))$,
- (ii) $H \in]\frac{1}{4}, \frac{1}{2}[$, $\lambda \in]0, 1[$ is such that $\lambda + H > \frac{1}{2}$, $\beta \in]0, 1[$, h is a measurable function defined on $[0, T]^{n+1}$ such that the mapping $t \mapsto h(\cdot, t)$ belongs to $\mathcal{C}^{\beta}([0, T]; \mathbb{H}^{\lambda}(\Lambda_T^{(n)}))$,

yields the following:

For any closed set $F \subset \mathcal{C}^{\gamma,0}([0, T])$,

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{P}^H \left(\mathbf{I}_h^{(n),H,\varepsilon} \in F \right) \leq -\mathcal{I}(F).$$

For any open set $G \subset \mathcal{C}^{\gamma,0}([0, T])$,

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{P}^H \left(\mathbf{I}_h^{(n),H,\varepsilon} \in G \right) \geq -\mathcal{I}(G).$$

Here $\gamma \in]0, \beta \wedge (H - 1/q)[$ under assumptions (i), $\gamma \in]0, \beta \wedge H[$, under (ii), respectively, and the rate functional \mathcal{I} is defined by

$$\mathcal{I}(\Phi) = \frac{\|\varphi\|_{\mathcal{H}^H}^2}{2}, \Phi \in \mathcal{C}^{\gamma,0}([0, T]),$$

if there exist $\varphi \in \mathcal{H}^H$ such that $\Phi = \mathbb{E} \left(\mathbf{I}_h^{(n),H}(B^H + \varphi) \right)$, and $\mathcal{I}(\Phi) = \infty$, otherwise.

The next discussion completes the description of the rate functional of the large deviation principle.

The construction of multiple integrals with respect to the fractional Brownian motion and their properties proved in the preceding sections have an analogue when the fractional Brownian motion is replaced by a deterministic function $\varphi \in \mathcal{H}^H$. More precisely, set

$$\mathbf{J}_t^{(n),H}(h) = \int_{\Lambda_t^{(n)}} h(\theta_1, \dots, \theta_n) \varphi(d\theta_1) \cdots \varphi(d\theta_n), \quad t \in [0, T],$$

then we have the following.

1. Let $H \in]\frac{1}{2}, 1[$ and $h \in L^{\frac{1}{H}}(\Lambda_T^{(n)})$. The function $t \rightarrow \mathbf{J}_t^{(n),H}(h)$ is well defined and satisfies

$$\sup_{t \in [0, T]} \left| \mathbf{J}_t^{(n),H}(h) \right| \leq C \|h\|_{L^{\frac{1}{H}}(\Lambda_T^{(n)})} \|\varphi\|_{\mathcal{H}^H}^n.$$

2. Let $H \in]\frac{1}{4}, \frac{1}{2}[$, $\lambda \in]0, 1[$ such that $\lambda + H > \frac{1}{2}$, and $h \in \mathbb{H}^\lambda(\Lambda_T^{(n)})$. The function $t \rightarrow \mathbf{J}_t^{(n),H}(h)$ is well defined and satisfies

$$\sup_{t \in [0, T]} \left| \mathbf{J}_t^{(n),H}(h) \right| \leq C \|h\|_{\mathbb{H}^\lambda(\Lambda_T^{(n)})} \|\varphi\|_{\mathcal{H}^H}^n.$$

Similarly, one can state the analogues of Corollaries 3.3 and 3.6 and prove the existence of $\mathbf{J}_h^{(n),H}$, that is, the function $t \rightarrow \mathbf{J}_t^{(n),H}(h(\cdot, t))$ of $\mathcal{C}^{\gamma,0}([0, T])$, for specific values of γ .

For the proof of these facts, we first establish recursively -as for the stochastic integrals- the representation

$$\mathbf{J}_t^{(n),H}(h) = \int_{[0,t]^n} \left(K_{H,t}^{*,(n)} h \right) (\theta_1, \dots, \theta_n) \dot{\varphi}(\theta_1) \cdots \dot{\varphi}(\theta_n) d\theta_1 \cdots d\theta_n.$$

Then, we apply Cauchy-Schwarz inequality and finally, (12) and (21) (with $s:=0$), respectively.

The stochastic integral $\mathbf{I}_h^{(1),H}(B^H + \varphi)$ satisfies $\mathbb{E} \left(\mathbf{I}_h^{(1),H}(B^H + \varphi) \right) = \mathbf{J}_h^{(1),H}$. Applying this fact recursively yields

$$\mathbb{E} \left(\mathbf{I}_h^{(n),H}(B^H + \varphi) \right) = \mathbf{J}_h^{(n),H}.$$

Hence, we recover the usual description of the rate functional in terms of the *skeleton* of the Gaussian functional.

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