Existence of size-minimizers in topological classes stable by localized homotopy

Vincent Feuvrier

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A minimal set — in the sense of Almgren — is a d-dimensional set in $\mathbf{R}^n$ ($0 \leq d < n$) such that its measure cannot be decreased by a deformation taken in a suitable class. In particular, we require our deformations to have compact support relatively to a given domain U. This prevents moving points that are close to the boundary of U, and allows considering the problem of finding minimal sets as a rewriting of the Plateau problem. In terms of currents, we want to minimize the size (measure of the support) and not the mass (integral of the multiplicity over the support). Mass-minimizing surfaces have been heavily studied in terms of differential geometry; in comparison, there are relatively few existence results for size-minimizers.

One of the technical difficulties of this setup is that the Hausdorff measure is generally not lower semi-continuous when taking limits — although the case $d = 1$ could be addressed using Gołąb's theorem. We propose an approach that is close in spirit to that of Reifenberg. It relies on a sequence of polyedric complexes adapted to a given minimizing sequence of the problem. We do a finite minimization over the d-dimensional faces of each complexe, and we show that by assuming suitable orientation and flatness properties of the polyhedrons, it is possible to build a minimizing sequence of sets that converges towards a minimal set for the Hausdorff metric convergence on every compact subset of U. It should be noted that this construction can be made in any dimensions d and n.

In some cases when $d = 2$ or $d = n - 1$, it is possible to conclude about the existence of a size minimizer in a given topological class stable by deformation. For instance, by using known regularity properties of Almgren-minimal sets such as Jean Taylor's theorem it is possible to build a retraction from a neighborhood of the limit onto itself — thus showing that the limit is still in the initial topological class.