Zabreiko's Result and its Consequences

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Let X be a normed space over the scalar field of either all real numbers or of all complex numbers. Let p be a seminorm on X, that is, (i) $p(x) \geq 0$ for all x in X, (ii) $p(x + y) \leq p(x) + p(y)$ for all x, y in X and (iii) $p(kx) = |k|p(x)$ for all x in X and all scalars k. It is easy to see that if p is continuous, then p is countably subadditive. In fact, if p is only lower semicontinuous, then p is countably subadditive. In 1936, Gelfand proved that if X is a Banach space and p is lower semicontinuous, then p is continuous. Generalizing this result, Zabreiko proved in 1969 that if X is a Banach space and p is countably subadditive, then p is continuous. Apparently this result of Zabreiko has remained largely unnoticed. It can be used to deduce several major theorems in Functional Analysis very easily.