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 t -NORM BASED FUZZY LOGICS

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Abstract

The study of the Gentzen system \mathcal{G}_{ew} determined by the well known sequent calculus \mathbf{FL}_{ew} [Ono98, Ono03c] is interesting for the study of the substructural aspects of t -norm based logics [Háj98, EG01]. In [BGV05] we studied the $\langle \vee, *, \neg, 0, 1 \rangle$ and the $\langle \vee, \wedge, *, \neg, 0, 1 \rangle$ -fragments of this Gentzen system and the same fragments of the logic of residuated lattices $IPC^* \setminus c$. In this paper we continue the study of the implication-less fragments of $IPC^* \setminus c$ and of \mathcal{G}_{ew} . We obtain the algebraization of the $\langle \vee, *, 0, 1 \rangle$, and the $\langle \vee, \wedge, *, 0, 1 \rangle$ -fragment of \mathcal{G}_{ew} . We obtain also that $\langle \vee, *, 0, 1 \rangle$ and the $\langle \vee, \wedge, *, 0, 1 \rangle$ -fragments of the logic of residuated lattices, $IPC^* \setminus c$, are exactly the $\langle \vee, \wedge, 0, 1 \rangle$ -fragment of classical logic. As a corollary of this fact we have that the $\langle \vee, *, 0, 1 \rangle$ and the $\langle \vee, \wedge, *, 0, 1 \rangle$ -fragments of every t -norm based logic are exactly the same fragments of classical logic.

Keywords: Substructural logics, t -norm based logics, residuated lattices, semilatticed monoids, Gentzen systems.

Introduction

Esteva and Godo defined [EG01] the logic MTL (Monoidal T -norm based Logic), a generalization of BL , the Basic Fuzzy Logic defined by Hájek [Háj98] and which is the logic

of continuous t-norms and their residua [CEG00]. It was proved in [JeMo02] that *MTL* is the logic of left-continuous t-norms and their residua, that is, the logic of the class of the (commutative integral bounded) residuated lattices defined in the real unit interval $[0, 1]$. The word ‘Monoidal’ in the name of this logic comes from the fact that *MTL* can be seen as an axiomatic extension of the so-called Monoidal Logic considered by Höhle in [Höh95], where a completeness theorem with respect to the class of all residuated lattices is proved. Monoidal Logic is definitionally equivalent to other systems considered in the literature: *H_{BCK}* [OK85] (corresponding to what Ono now calls **FL_{ew}**-logic), *IPC** \ *c* [AV00] (intuitionistic logic without contraction), etc. Beyond the connection of Monoidal Logic with residuated lattices by means a completeness theorem, we can also say that Monoidal Logic is the logic of residuated lattices in a stronger sense: it was proved in [AV00] that this logic is algebraizable in the sense of [BP89] and the variety of residuated lattices is its equivalent algebraic semantics. In [Go01] (see also [GGCB03]) it was shown that *MTL* can be obtained as the extension of the Monoidal Logic with the *prelinearity* axiom $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$. Thus, the study of the logic of residuated lattices is important in the context of the studies of the t-norm based fuzzy logics [Háj98] because it is a subsystem of the logic of left-continuous t-norms *MTL* and so it is a subsystem of every t-norm based logic (for a survey of residuated t-norm based fuzzy logics see [EGG03]).

It was shown in [AV00] that the Monoidal Logic *IPC** \ *c* is associated in a natural and strong sense to the Gentzen system (which we denote here by \mathcal{G}_{ew}) obtained from the well known sequent calculus **FL_{ew}** [OK85]: \mathcal{G}_{ew} is equivalent to its external deductive system¹, which is exactly the system *IPC** \ *c*. It was also proved in [Ad01] that the fragments of \mathcal{G}_{ew} containing the implication connective are equivalent to the corresponding fragments of *IPC** \ *c*, and also their equivalent algebraic semantics were obtained. The study of the implication-less fragments² of \mathcal{G}_{ew} and of *IPC** \ *c* started in [BGV05]. The calculi considered in [BGV05] are **FL_{ew}**[$\vee, *, \neg$], obtained from **FL_{ew}** by dropping the rules of introduction for the connectives of additive conjunction and implication, and **FL_{ew}**[$\vee, \wedge, *, \neg$], obtained from **FL_{ew}** by dropping the rules of introduction for the connectives of implication. It was proved in [BGV05] that the Gentzen systems associated to these calculi are algebraizable and the varieties of (commutative integral bounded) pseudocomplemented semilatticed and latticed monoids are their equivalent algebraic semantics, respectively. Moreover, it was also proved that these Gentzen systems are the fragments of \mathcal{G}_{ew} in the adequate languages. It is important to stress that, in contrast with what happens in the cases with implication, these Gentzen systems are not equivalent to their external deductive systems. These external deductive systems are fragments of *IPC** \ *c*, are decidable, and the varieties of pseudocomplemented semilatticed and latticed monoids are algebraic semantics (in the sense of [BP89]) respectively of these systems, with defining equation $p \approx 1$. These deductive systems are strictly included in the corresponding fragments of intuitionistic logic.

In this paper we continue the study of the implication-less fragments of \mathcal{G}_{ew} and *IPC** \ *c*. It was proved in [AV02] that the systems without implication, denoted in the present paper by $\mathcal{G}_{\text{ew}}[\vee, *]$ and $\mathcal{G}_{\text{ew}}[\vee, \wedge, *]$, are algebraizable and that the varieties of (commutative integral bounded) semilatticed monoids and latticed monoids are their algebraic semantics respectively. We prove (Corollary 4.6) that $\mathcal{G}_{\text{ew}}[\vee, *]$ and $\mathcal{G}_{\text{ew}}[\vee, \wedge, *]$ are fragments of \mathcal{G}_{ew} and (Corollary 4.10) that these fragments are decidable. We also prove (Corollary 4.11) that the external systems of $\mathcal{G}_{\text{ew}}[\vee, *]$ and $\mathcal{G}_{\text{ew}}[\vee, \wedge, *]$, that is, the systems $\mathcal{E}_{\text{ew}}[\vee, *]$ and

¹For the notion of equivalence used here and the notion of *external deductive system* associated to a Gentzen system see Section 1.

²We stress that our notion of fragment considers the full consequence relation admitting hypotheses.

$\mathcal{E}_{\text{ew}}[\vee, \wedge, *]$, are fragments of $IPC^* \setminus c$. Finally, by using the property that in any subdirectly irreducible residuated lattice, if $x \vee y = 1$ then either $x = 1$ or $y = 1$ we obtain (Theorem 5.2) that the $\langle \vee, *, 0, 1 \rangle$ and the $\langle \vee, \wedge, *, 0, 1 \rangle$ -fragments of $IPC^* \setminus c$ are exactly the $\langle \vee, \wedge, 0, 1 \rangle$ -fragment of classical logic. As an immediate consequence of these results we obtain (Corollary 5.3) that the $\langle \vee, *, 0, 1 \rangle$ and the $\langle \vee, \wedge, *, 0, 1 \rangle$ -fragments of every t-norm based logic are the $\langle \wedge, \vee, 0, 1 \rangle$ -fragment of classical logic. Finally, we give an alternative axiomatization of the $\langle \vee, *, 0, 1 \rangle$ -fragment of classical logic by means a sequent calculus without contraction (Corollary 5.4).

1 Basic concepts

The logical systems that we consider in this paper are Gentzen systems and deductive systems, the latter being a particular case of the former. Most of the literature on Gentzen systems, and on deductive systems, focuses only on their derivable sequents, i.e., on the sequents derivable without any hypothesis. Our approach is completely different since we analyze the full consequence relation admitting hypotheses in the proofs. The reader should bear in mind this difference between our approach and the one commonly considered in the literature. In this section we introduce and clarify, from this more general perspective, the notions that we will need later. Let us recall the notions concerning Gentzen systems and their algebraization that will appear in the following sections. The results in this section are presented without proofs; the reader interested in the proofs can check [RV93, RV95, GTV97].

Gentzen systems. By a *propositional language* we mean an algebraic signature. Given a propositional language \mathcal{L} we will denote by $Fm_{\mathcal{L}}$ the set of \mathcal{L} -formulas and by $\mathbf{Fm}_{\mathcal{L}}$ the algebra of \mathcal{L} -formulas. Throughout the paper, we will follow the convention of using boldface for algebras. We will use the lowercase letters φ, ψ, \dots for \mathcal{L} -formulas, and the uppercase Γ, Δ, \dots for finite (maybe empty) sequences of \mathcal{L} -formulas. Given $m, n \in \omega$, an \mathcal{L} -*sequent of type* $\langle m, n \rangle$ is a pair $\varsigma = \langle \Gamma, \Delta \rangle$ of finite sequences of \mathcal{L} -formulas such that the length of Γ is m and the length of Δ is n . While ς will refer to a \mathcal{L} -sequent, we will use the metavariable Φ for sets of \mathcal{L} -sequents. We will write \emptyset for the empty sequence³, φ for $\langle \varphi \rangle$, $\Gamma \Rightarrow \Delta$ for the sequent $\langle \Gamma, \Delta \rangle$, and $\varphi_0, \dots, \varphi_{m-1} \Rightarrow \psi_0, \dots, \psi_{n-1}$ instead of $\langle \varphi_0, \dots, \varphi_{m-1} \rangle \Rightarrow \langle \psi_0, \dots, \psi_{n-1} \rangle$. Given a set $\mathcal{T} \subseteq \omega \times \omega$ we will denote by $Seq_{\mathcal{L}}^{\mathcal{T}}$ the set of all \mathcal{L} -sequents with type belonging to \mathcal{T} .

A *Gentzen system* is a triple $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash \rangle$ where \mathcal{L} is a propositional language, \mathcal{T} is a non-empty set of pairs of natural numbers, and \vdash is a relation between subsets of $Seq_{\mathcal{L}}^{\mathcal{T}}$ and elements of $Seq_{\mathcal{L}}^{\mathcal{T}}$ satisfying the following conditions.

- 1) If $\varsigma \in \Phi$, then $\Phi \vdash \varsigma$.
- 2) If $\Phi \vdash \varsigma$ and for every $\varsigma' \in \Phi$, $\Phi' \vdash \varsigma'$, then $\Phi' \vdash \varsigma$.
- 3) If $\Phi \vdash \varsigma$ and $\Phi \subseteq \Phi'$, then $\Phi' \vdash \varsigma$.
- 4) If $\Phi \vdash \varsigma$, then $e[\Phi] \vdash e(\varsigma)$ for any substitution e (i.e., for any endomorphism of the algebra $\mathbf{Fm}_{\mathcal{L}}$)⁴.

³The context will tell us if this symbol denotes the empty set or the empty sequence.

⁴Here $e(\varphi_0, \dots, \varphi_{m-1} \Rightarrow \psi_0, \dots, \psi_{n-1})$ is obviously defined as the sequent $e(\varphi_0), \dots, e(\varphi_{m-1}) \Rightarrow e(\psi_0), \dots, e(\psi_{n-1})$.

The first three conditions say that \vdash is a *consequence relation* or a *closure operator* on the set $Seq_{\mathcal{L}}^{\mathcal{T}}$, and the last one is called *invariance under substitutions*. The Gentzen system is *finitary* if, moreover, it satisfies the following condition:

- 5) If $\Phi \vdash \varsigma$, then there is a finite subset Φ' of Φ with $\Phi' \vdash \varsigma$.

For the sake of simplicity, we will only consider finitary Gentzen systems. Thus, we will refer to finitary Gentzen systems simply as Gentzen systems. A well known way to define a Gentzen system is through derivations in a *sequent calculus*. As usual, we will write $\Phi, \varsigma \vdash \varsigma'$ instead of $\Phi \cup \{\varsigma\} \vdash \varsigma'$. The set \mathcal{T} is called the *type* of \mathcal{G} . The components of a Gentzen system \mathcal{G} will sometimes be written respectively as $\mathcal{L}(\mathcal{G})$, $\mathcal{T}(\mathcal{G})$ and $\vdash_{\mathcal{G}}$ since this avoids any ambiguity. Two sequents ς and ς' are *\mathcal{G} -equivalent* (notation: $\varsigma \dashv\vdash_{\mathcal{G}} \varsigma'$ or simply $\varsigma \dashv\vdash \varsigma'$) if it holds at the same time that $\varsigma \vdash_{\mathcal{G}} \varsigma'$ and that $\varsigma' \vdash_{\mathcal{G}} \varsigma$. A sequent ς is said *\mathcal{G} -derivable* if $\emptyset \vdash_{\mathcal{G}} \varsigma$.

The definition of Gentzen system generalizes the notion of deductive system defined by Blok and Pigozzi in [BP89]. It turns out that a *deductive system* \mathcal{S} is no less than a Gentzen system with type $\{0\} \times \{1\}$ where the formula φ is identified with the sequent $\emptyset \Rightarrow \varphi$.

Fragments. Let \mathcal{G} be a Gentzen system $\langle \mathcal{L}, \mathcal{T}, \vdash_{\mathcal{G}} \rangle$, and let \mathcal{L}' be a sublanguage of \mathcal{L} . The *\mathcal{L}' -fragment* of \mathcal{G} is the Gentzen system $\mathcal{G}' = \langle \mathcal{L}', \mathcal{T}, \vdash_{\mathcal{G}'} \rangle$ defined by the fact that for all $\Phi \cup \{\varsigma\} \subseteq Seq_{\mathcal{L}'}^{\mathcal{T}}$,

$$\Phi \vdash_{\mathcal{G}'} \varsigma \quad \text{iff} \quad \Phi \vdash_{\mathcal{G}} \varsigma.$$

In this case it is said that \mathcal{G} is a *conservative expansion* of \mathcal{G}' . We stress that this notion of fragment considers the full consequence relation and not just the derivable sequents.

Algebraization of Gentzen systems. A class \mathbf{K} of \mathcal{L} -algebras is an *algebraic semantics* for a Gentzen system $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash \rangle$ in the case that there is a translation $\tau : Seq_{\mathcal{L}}^{\mathcal{T}} \longrightarrow \mathcal{P}_{fin}(Eq_{\mathcal{L}})$ such that for all $\Phi \cup \{\varsigma\} \subseteq Seq_{\mathcal{L}}^{\mathcal{T}}$,

$$\Phi \vdash \varsigma \quad \text{iff} \quad \tau[\Phi] \models_{\mathbf{K}} \tau(\varsigma).$$

If moreover there is a kind of inverse translation then what we obtain is the notion of algebraization. To be more precise, a Gentzen system \mathcal{G} is said to be *algebraizable with equivalent algebraic semantics* \mathbf{K} if there is a translation $\tau : Seq_{\mathcal{L}}^{\mathcal{T}} \longrightarrow \mathcal{P}_{fin}(Eq_{\mathcal{L}})$ and a translation $\rho : Eq_{\mathcal{L}} \longrightarrow \mathcal{P}_{fin}(Seq_{\mathcal{L}}^{\mathcal{T}})$ such that

- 1) for all $\Phi \cup \{\varsigma\} \subseteq Seq_{\mathcal{L}}^{\mathcal{T}}$, it holds that $\Phi \vdash \varsigma$ iff $\tau[\Phi] \models_{\mathbf{K}} \tau(\varsigma)$,
- 2) for all $\Phi \cup \{\varsigma\} \subseteq Eq_{\mathcal{L}}$, it holds that $\Phi \models_{\mathbf{K}} \varsigma$ iff $\rho[\Phi] \vdash \rho(\varsigma)$,
- 3) for all $\varsigma \in Eq_{\mathcal{L}}$, it holds that $\varsigma \models_{\mathbf{K}} \tau\rho(\varsigma)$,
- 4) for all $\varsigma \in Seq_{\mathcal{L}}^{\mathcal{T}}$, it holds that $\varsigma \dashv\vdash \rho\tau(\varsigma)$.

If we replace $Eq_{\mathcal{L}}$ with $Seq_{\mathcal{L}}^{\mathcal{T}}$ in the previous definition what we obtain is the more general notion of *equivalence between Gentzen systems*. Notice that if \mathbf{K} is a class of \mathcal{L} -algebras then the equational logic $\models_{\mathbf{K}}$ can be seen as a Gentzen system with language \mathcal{L} and type $\{1\} \times \{1\}$ where we identify an equation $\varphi \approx \psi \in Eq_{\mathcal{L}}$ with the sequent $\varphi \Rightarrow \psi$. Thus the algebraization of a Gentzen system can be seen as a particular case of equivalence between Gentzen systems. It is well known that the definition of equivalence is redundant because

the conjunction of 1) and 3) is equivalent to the conjunction of 2) and 4) [RV95, Proposition 2.1].

It holds that if K is an equivalent algebraic semantics for \mathcal{G} , then so is the quasivariety K^Q generated by K [GTV97, Corollary 4.2]. It is also known that if K and K' are equivalent algebraic semantics for \mathcal{G} , then K and K' generates the same quasivariety [GTV97, Corollary 4.4]. This quasivariety is called *the equivalent quasivariety semantics* for \mathcal{G} .

We notice that if \mathcal{S} is a deductive system then the fact that it is algebraizable in the sense of [BP89] with the set of equivalence formulas $\Delta(p, q)$ and the set of defining equations $\Theta(p)$ coincides precisely with the fact of being algebraizable in the above sense under the translations $\tau(p) := \Theta(p)$ and $\rho(p \approx q) := \Delta(p, q)$. Hence, the algebraization of Gentzen systems generalizes the algebraization of deductive systems introduced in [BP89].

External deductive system associated with a Gentzen system. Let \mathcal{G} be a Gentzen system $\langle \mathcal{L}, \mathcal{T}, \vdash \rangle$. There are at least two methods in the literature used to associate a deductive system with \mathcal{G} . The common method is based on considering the derivable sequents. Specifically, $\Sigma \vdash_{\mathcal{I}(\mathcal{G})} \varphi$ holds when

there is a finite subset $\{\varphi_0, \dots, \varphi_{n-1}\}$ of Σ such that $\emptyset \vdash \varphi_0, \dots, \varphi_{n-1} \Rightarrow \varphi$.

We notice that this approach yields a deductive system, called *internal*, only if the Gentzen system satisfies some of the structural rules. Another method, which works for all Gentzen systems such that $\langle 0, 1 \rangle \in \mathcal{T}$ (even when structural rules are not satisfied), yields the external deductive system⁵. The *external deductive system* associated with \mathcal{G} is defined as the deductive system $\mathcal{E}(\mathcal{G})$ such that $\Sigma \vdash_{\mathcal{E}(\mathcal{G})} \varphi$ iff

$$\{\emptyset \Rightarrow \psi : \psi \in \Sigma\} \vdash \emptyset \Rightarrow \varphi.$$

Since we have restricted ourselves to finitary Gentzen systems it is clear that $\mathcal{E}(\mathcal{G})$ is finitary.

2 The logical systems that we study

In this section we recall some logical systems considered in the literature concerning intuitionistic logic without contraction and we introduce the logical systems that we study in this paper. First of all let us recall the definition of the well known sequent calculus $\mathbf{FL}_{\mathbf{ew}}$ given in the language $\langle \vee, \wedge, *, \rightarrow, 0, 1 \rangle$.

Definition 2.1. (Cf. [Ono98], [Ono03c]) Let $\mathcal{L} = \langle \vee, \wedge, *, \rightarrow, 0, 1 \rangle$ be a propositional language of type $\langle 2, 2, 2, 2, 0, 0 \rangle$. Let φ, ψ, ξ be \mathcal{L} -formulas; Γ, Π finite sequences (possibly empty) of \mathcal{L} -formulas. $\mathbf{FL}_{\mathbf{ew}}$ is the calculus of \mathcal{L} -sequents of type $\omega \times \{1\}$ defined by the following axioms and rules⁶:

Axioms:

$$\varphi \Rightarrow \varphi \quad (\text{Axiom 1}) \quad 0 \Rightarrow \varphi \quad (\text{Axiom 2}) \quad \emptyset \Rightarrow 1 \quad (\text{Axiom 3})$$

Structural rules:

$$\frac{\Gamma \Rightarrow \varphi \quad \varphi, \Pi \Rightarrow \xi}{\Gamma, \Pi \Rightarrow \xi} \quad (\text{Cut})$$

⁵We use the words 'internal' and 'external' following Avron (see [Avr88]).

⁶Strictly speaking each of these rules is a family of rules, and not only a rule.

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \xi}{\Gamma, \psi, \varphi, \Pi \Rightarrow \xi} (e \Rightarrow) \quad \frac{\Gamma \Rightarrow \xi}{\varphi, \Gamma \Rightarrow \xi} (w \Rightarrow)$$

Rules of introduction of connectives:

$$\frac{\varphi, \Gamma \Rightarrow \xi \quad \psi, \Gamma \Rightarrow \xi}{\varphi \vee \psi, \Gamma \Rightarrow \xi} (\vee \Rightarrow) \quad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} (\Rightarrow \vee_1) \quad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi} (\Rightarrow \vee_2)$$

$$\frac{\varphi, \Gamma \Rightarrow \xi}{\varphi \wedge \psi, \Gamma \Rightarrow \xi} (\wedge_1 \Rightarrow) \quad \frac{\psi, \Gamma \Rightarrow \xi}{\varphi \wedge \psi, \Gamma \Rightarrow \xi} (\wedge_2 \Rightarrow) \quad \frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi} (\Rightarrow \wedge)$$

$$\frac{\varphi, \psi, \Gamma \Rightarrow \xi}{\varphi * \psi, \Gamma \Rightarrow \xi} (* \Rightarrow) \quad \frac{\Gamma \Rightarrow \varphi \quad \Pi \Rightarrow \psi}{\Gamma, \Pi \Rightarrow \varphi * \psi} (\Rightarrow *)$$

$$\frac{\Gamma \Rightarrow \varphi \quad \psi, \Pi \Rightarrow \xi}{\Gamma, \varphi \rightarrow \psi, \Pi \Rightarrow \xi} (\rightarrow \Rightarrow) \quad \frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightarrow \psi} (\Rightarrow \rightarrow)$$

The Gentzen system associated with \mathbf{FL}_{ew} will be denoted by \mathcal{G}_{ew} .

Remark 2.2. When we add the structural rule of contraction

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} (c \Rightarrow)$$

to the previous calculus what we obtain is \mathbf{FL}_{ewc} [Ono98, Ono03c], which is a redundant version of the Gentzen's calculus LJ for the intuitionistic propositional logic since the multiplicative conjunction $*$ behaves as the additive conjunction \wedge .

Theorem 2.3. (Cf. [Ono98, Theorem 6]) *Cut elimination holds for \mathbf{FL}_{ewc} and \mathbf{FL}_{ew} .*

Next we recall the definition of the deductive system $IPC^* \setminus c$. This system has been studied in slightly different languages from the one that we take, since $IPC^* \setminus c$ is definitionally equivalent to H_{BCK} [OK85], to the monoidal logic [Höh95] and to the systems introduced with the same name in [AV00] and [BGV05].

Definition 2.4. $IPC^* \setminus c$ is the deductive system in the language $\mathcal{L} = \langle \vee, \wedge, *, \rightarrow, 0, 1 \rangle$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$, defined by the Modus Ponens rule and the following axioms (using implication as the least binding connective):

- | | | |
|-------|---|-----|
| (A1) | $(\varphi \rightarrow \psi) \rightarrow ((\gamma \rightarrow \varphi) \rightarrow (\gamma \rightarrow \psi))$ | (B) |
| (A2) | $(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \gamma))$ | (C) |
| (A3) | $\varphi \rightarrow (\psi \rightarrow \varphi)$ | (K) |
| (A4) | $(\varphi \rightarrow \gamma) \rightarrow ((\psi \rightarrow \gamma) \rightarrow (\varphi \vee \psi \rightarrow \gamma))$ | |
| (A5) | $\varphi \rightarrow \varphi \vee \psi$ | |
| (A6) | $\psi \rightarrow \varphi \vee \psi$ | |
| (A7) | $\varphi \wedge \psi \rightarrow \varphi$ | |
| (A8) | $\varphi \wedge \psi \rightarrow \psi$ | |
| (A9) | $\varphi \rightarrow (\psi \rightarrow \varphi \wedge \psi)$ | |
| (A10) | $(\gamma \rightarrow \varphi) \wedge (\gamma \rightarrow \psi) \rightarrow (\gamma \rightarrow \varphi \wedge \psi)$ | |
| (A11) | $\varphi \rightarrow (\psi \rightarrow \varphi * \psi)$ | |
| (A12) | $(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow (\varphi * \psi \rightarrow \gamma)$ | |
| (A13) | $0 \rightarrow \varphi$ | |
| (A14) | $\varphi \rightarrow 1$ | |

In [AV00] it was proved that $\mathcal{G}_{\mathbf{ew}}$, called there $\mathcal{G}_{LJ^*\setminus c}$, and $IPC^*\setminus c$ are equivalent as Gentzen systems. Let us recall this result.

Theorem 2.5. (Cf. [AV00, Theorem 11]) $\mathcal{G}_{\mathbf{ew}}$ and $IPC^*\setminus c$ are equivalent, with translations τ and ρ defined as follows:

$$\tau(\varphi_0, \dots, \varphi_{m-1} \Rightarrow \varphi) := \begin{cases} \{\varphi_0 \rightarrow (\varphi_1 \rightarrow (\dots \rightarrow (\varphi_{m-1} \rightarrow \varphi) \dots))\}, & \text{if } m \geq 1 \\ \{\varphi\}, & \text{if } m = 0 \end{cases}$$

$$\rho(\varphi) := \{\emptyset \Rightarrow \varphi\}.$$

That is, the following conditions are satisfied:

- (1) For every $\Sigma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Sigma \vdash_{IPC^*\setminus c} \varphi$ iff $\{\rho(\sigma) : \sigma \in \Sigma\} \vdash_{\mathbf{FL}_{\mathbf{ew}}} \rho(\varphi)$.
- (2) For every $\Gamma \Rightarrow \varphi \in Seq_{\mathcal{L}}^{\omega \times \{1\}}$, $\Gamma \Rightarrow \varphi \dashv\vdash_{\mathbf{FL}_{\mathbf{ew}}} \rho\tau(\Gamma \Rightarrow \varphi)$.

Next we will see that $IPC^*\setminus c$ is the external deductive system associated to $\mathcal{G}_{\mathbf{ew}}$.

Corollary 2.6. $IPC^*\setminus c$ is the external deductive system of $\mathcal{G}_{\mathbf{ew}}$. Then, $\mathcal{G}_{\mathbf{ew}}$ is equivalent to its associated external deductive system.

Proof: We will denote by $\mathcal{E}_{\mathbf{ew}}$ the external deductive system associated to $\mathcal{G}_{\mathbf{ew}}$. By (1) of Theorem 2.5, the definition of the translation ρ and the definition of the external deductive system we have

$$\begin{aligned} \Sigma \vdash_{IPC^*\setminus c} \varphi & \text{ iff } \{\rho(\sigma) : \sigma \in \Sigma\} \vdash_{\mathbf{FL}_{\mathbf{ew}}} \rho(\varphi) & \text{ iff} \\ \{\emptyset \Rightarrow \sigma : \sigma \in \Sigma\} \vdash_{\mathbf{FL}_{\mathbf{ew}}} \emptyset \Rightarrow \varphi & \text{ iff } \Sigma \vdash_{\mathcal{E}_{\mathbf{ew}}} \varphi. \end{aligned}$$

□

Now we introduce the logical systems which are the aim of this paper.

Definition 2.7 (The calculi).

- $\mathbf{FL}_{\mathbf{ew}}[\vee, *]$ is the sequent calculus in the language $\langle \vee, *, 0, 1 \rangle$ obtained by deleting from $\mathbf{FL}_{\mathbf{ew}}$ the rules of introduction of the additive conjunction and the implication, i.e., we consider all their axioms, all their structural rules and their introduction rules simply for the connectives $\vee, *$.
- $\mathbf{FL}_{\mathbf{ew}}[\vee, \wedge, *]$ is the sequent calculus in the language $\langle \vee, \wedge, *, 0, 1 \rangle$ obtained as before except for the fact that we also consider the introduction rules for \wedge .
- $\mathbf{FL}_{\mathbf{ewc}}[\vee, *]$ is the sequent calculus in the language $\langle \vee, *, 0, 1 \rangle$ obtained by adding to $\mathbf{FL}_{\mathbf{ew}}[\vee, *]$ the structural rule of contraction (see Remark 2.8).

Remark 2.8. It is easy to see that the following rules are derived from the calculus $\mathbf{FL}_{\mathbf{ewc}}[\vee, *]$:

$$\frac{\varphi, \Gamma \Rightarrow \xi}{\varphi * \psi, \Gamma \Rightarrow \xi} \quad \frac{\psi, \Gamma \Rightarrow \xi}{\varphi * \psi, \Gamma \Rightarrow \xi} \quad \frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi * \psi}$$

Thus, in presence of contraction, the multiplicative conjunction behaves as the additive one. Therefore, the Gentzen system determined by $\mathbf{FL}_{\mathbf{ewc}}[\vee, *]$ is equal to the Gentzen system determined by the calculus, say $\mathbf{FL}_{\mathbf{ewc}}[\vee, \wedge]$, in the language $\langle \vee, \wedge, 0, 1 \rangle$ which has the same axioms and structural rules as $\mathbf{FL}_{\mathbf{ewc}}$ and as logical rules the rules of introduction for the additive connectives. Note that the only difference between the two calculi is in notation, since in $\mathbf{FL}_{\mathbf{ewc}}[\vee, *]$ we denote by the symbol $*$ the additive conjunction.

Theorem 2.9. *Cut elimination holds for $\mathbf{FL}_{\mathbf{ew}}[\vee, *]$, $\mathbf{FL}_{\mathbf{ew}}[\vee, \wedge, *]$ and $\mathbf{FL}_{\mathbf{ewc}}[\vee, *]$.*

Proof. It is an immediate consequence of Theorem 2.3. \square

Definition 2.10 (The Gentzen systems and their external systems). The Gentzen system determined by the calculi $\mathbf{FL}_{\mathbf{ew}}[\vee, *]$, $\mathbf{FL}_{\mathbf{ew}}[\vee, \wedge, *]$ and $\mathbf{FL}_{\mathbf{ewc}}[\vee, *]$ are denoted by $\mathcal{G}_{\mathbf{ew}}[\vee, *]$, and $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *]$, and $\mathcal{G}_{\mathbf{ewc}}[\vee, *]$, respectively. The external deductive systems associated will be denoted $\mathcal{E}_{\mathbf{ew}}[\vee, *]$, and $\mathcal{E}_{\mathbf{ew}}[\vee, \wedge, *]$, and $\mathcal{E}_{\mathbf{ewc}}[\vee, *]$, respectively.

Definition 2.11 (The fragments). Let Ψ be one of the languages $\langle \vee, *, 0, 1 \rangle$, $\langle \vee, \wedge, *, 0, 1 \rangle$ or $\langle \vee, \wedge, 0, 1 \rangle$. And let \mathcal{S} any deductive system or, in general, any Gentzen system. The Ψ -fragment of \mathcal{S} will be denoted by $\Psi\text{-}\mathcal{S}$.

In Section 4 we will prove the following facts:

- The Gentzen systems $\mathcal{G}_{\mathbf{ew}}[\vee, *]$ and $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *]$ are fragments of $\mathcal{G}_{\mathbf{ew}}$, that is, are equal to $\langle \vee, *, 0, 1 \rangle\text{-}\mathcal{G}_{\mathbf{ew}}$ and $\langle \vee, \wedge, *, 0, 1 \rangle\text{-}\mathcal{G}_{\mathbf{ew}}$, respectively.
- The deductive systems $\mathcal{E}_{\mathbf{ew}}[\vee, *]$ and $\mathcal{E}_{\mathbf{ew}}[\vee, \wedge, *]$ are fragments of $IPC^*\setminus c$, that is, are equal to $\langle \vee, *, 0, 1 \rangle\text{-}IPC^*\setminus c$ and $\langle \vee, \wedge, *, 0, 1 \rangle\text{-}IPC^*\setminus c$, respectively.

Moreover, in Section 5 we show that the fragments of $IPC^*\setminus c$ in the languages $\langle \vee, *, 0, 1 \rangle$, $\langle \vee, \wedge, *, 0, 1 \rangle$ and $\langle \vee, \wedge, 0, 1 \rangle$ are essentially the same logic that the $\langle \vee, \wedge, 0, 1 \rangle$ -fragment of the classical propositional logic.

Remark 2.12. Again we stress that our notion of fragment also considers the proofs admitting hypotheses, and not just the proofs without hypotheses. For instance, for the case of $\mathcal{G}_{\mathbf{ew}}[\vee, *]$ this means that

$$\Phi \vdash_{\mathbf{FL}_{\mathbf{ew}}[\vee, *]} \varsigma \quad \text{iff} \quad \Phi \vdash_{\mathbf{FL}_{\mathbf{ew}}} \varsigma. \quad (1)$$

for every set $\Phi \cup \{\varsigma\}$ of sequents in the language $\langle \vee, *, 0, 1 \rangle$. Indeed, from the following theorem it trivially follows that (1) holds for the previous two Gentzen systems when $\Phi = \emptyset$.

Theorem 2.13. *Cut elimination holds for $\mathbf{FL}_{\mathbf{ew}}[\vee, *]$ and $\mathbf{FL}_{\mathbf{ew}}[\vee, \wedge, *]$.*

Proof. It is an immediate consequence of Theorem 2.3. \square

3 The associated algebraic counterpart

In this section we introduce the algebraic structures needed to study the logics developed in the paper, and prove several facts about them. First of all we recall the definition of the class of residuated lattices.⁷

The class \mathbb{RL} of *residuated lattices* is the class of algebras $\mathbf{A} = \langle A, \vee, \wedge, *, \rightarrow, 0, 1 \rangle$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ satisfying the following conditions:

- 1) $\langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded lattice with associated order \leq ,

⁷We stress that in the more recent literature, e.g. [JT02, Ono03a, Ono03c], these algebras are sometimes called *commutative integral bounded residuated lattices* to distinguish them from the non-commutative and non-integral case.

- 2) $\langle A, *, 1 \rangle$ is a commutative monoid with the unit 1,
- 3) $x * z \leq y \Leftrightarrow z \leq x \rightarrow y$ (the law of *residuation*),

The law of residuation implies the following distributivity of $*$ over \vee :

$$a * \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a * b_i). \quad (2)$$

This law must be read as saying that if $\{a, b_i\}_{i \in I} \subseteq A$ and $\bigvee_{i \in I} b_i$ exists, then $\bigvee_{i \in I} (a * b_i)$ also exists and the previous equality holds. It is known that \mathbb{RL} can be axiomatized using only equations, that is, \mathbb{RL} is a variety. For more information about residuated lattices see [KO00, KO01, BvA02, Ono03a, Ono03b].

Definition 3.1. (Cf [Ono03a]) An algebra $\mathbf{A} = \langle A, \vee, *, 0, 1 \rangle$ of type $(2, 2, 0, 0)$ is a *semi-latticed monoid* (*sl-monoid*) if it satisfies:

- 1) $\langle A, \vee, 0, 1 \rangle$ is a bounded semilattice,
- 2) $\langle A, *, 1 \rangle$ is a commutative monoid with unit 1,
- 3) $(x \vee y) * z \approx (x * z) \vee (y * z)$.

The class of semilatticed monoids is denoted by \mathbb{M}^{sl} . An algebra $\mathbf{A} = \langle A, \vee, \wedge, *, 0, 1 \rangle$ of type $(2, 2, 2, 0, 0)$ such that the reduct is a semilatticed monoid and $\langle A, \vee, \wedge \rangle$ is a lattice is called a *latticed monoid* (*l-monoid*). The class of latticed monoids is denoted by \mathbb{M}^l .

A more accurate name for these algebras would include the words *commutative integral bounded* at the beginning, but for the sake of simplicity in the typography we adopt the nomenclature given in the definition. Obviously, these classes are varieties and in them it holds that i) $*$ is monotone in both arguments, ii) $x * y \leq x$ and $x * y \leq y$, and iii) $x * 0 \approx 0$.

Proposition 3.2. Let $\mathbf{A} \in \mathbb{M}^{sl}$. The following conditions are equivalent:

- i) $\mathbf{A} \models x * x \approx x$
- ii) For every $a, b \in A$, $a * b = \text{Inf}\{a, b\}$
- iii) $\mathbf{A} \models x \leq y \Leftrightarrow x * y \approx x$.

Proof: Let $a, b, c \in A$.

$i) \Rightarrow ii)$: As \mathbf{A} is integral we have that $a * b$ is a lower bound of $\{a, b\}$. Let c be a lower bound of $\{a, b\}$. Then, by monotonicity, $c * c \leq a * b$. So, by ii), we have $c \leq a * b$.

$ii) \Rightarrow iii)$ and $iii) \Rightarrow i)$ are trivial. □

Proposition 3.3. The variety of bounded distributive lattices is the subvariety of semilatticed monoids defined by the equation $x * x \approx x$, i.e.,

$$\begin{aligned} \{\mathbf{A} : \mathbf{A} = \langle A, \vee, *, 0, 1 \rangle \text{ bounded distributive lattice}\} = \\ = \{\mathbf{A} : \mathbf{A} = \langle A, \vee, *, 0, 1 \rangle \text{ semilatticed monoid and } \mathbf{A} \models x * x \approx x\}. \end{aligned}$$

Proof: It is a consequence of $i) \Rightarrow ii)$ of Proposition 3.2. □

Remark 3.4. From the above statement we can consider the (commutative, integral, bounded) semilatticed and latticed monoids as generalizations of bounded distributive lattices. Roughly speaking, as we will see next, the behaviour of semilatticed monoids with respect to residuated lattices is the same one as the behaviour of bounded distributive lattices with respect to Heyting algebras. For instance, Theorem 3.11 generalizes the well-known fact that every bounded distributive lattice is the subreduct of a Heyting algebra, which is a particular case of Theorem 3.11 using the fact that the embedding preserves all the existing meets.

Since the fact that every residuated lattice satisfies the distributivity of the monoidal operation with respect to the operation \vee , we have that the reducts of every residuated lattice in the adequate languages are, respectively, in \mathbb{M}^{sl} and in \mathbb{M}^ℓ . But are all the \mathbb{M}^{sl} -algebras and \mathbb{M}^ℓ -algebras the reduct of a residuated lattice? In what follows we will see that this is true for finite \mathbb{M}^{sl} -algebras and \mathbb{M}^ℓ -algebras and also for the algebras of this varieties satisfying the infinite distributivity (2). Nevertheless we will show that for the general case the answer is negative.

Definition 3.5. We will say that a \mathbb{M}^{sl} -algebra is *complete* if it is a complete semilattice as an ordered set, and that a \mathbb{M}^ℓ -algebra is *complete* if it is a complete lattice as an ordered set.

Proposition 3.6. *Every complete \mathbb{M}^{sl} -algebra is the $\langle \vee, *, 0, 1 \rangle$ -reduct of a complete \mathbb{M}^ℓ -algebra.*

Proof: Let \mathbf{A} be a complete \mathbb{M}^{sl} -algebra. Since A has a minimum element, then we have that the ordered set $\langle A, \leq \rangle$ associated to the complete semilattice is a complete lattice such that, for every subset $X \subseteq A$, it holds that $\bigwedge X = \bigvee X^\leftarrow$, i.e., $\max X^\leftarrow = \bigvee X^\leftarrow$. So, \mathbf{A} is the $\langle \vee, *, 0, 1 \rangle$ -reduct of the complete \mathbb{M}^ℓ -algebra of universe A in such a way that the operation \wedge is defined by $a \wedge b =: \bigvee \{x \in A : x \leq a \text{ and } x \leq b\}$ and the monoidal operation is the one of \mathbf{A} . \square

Proposition 3.7. *Let \mathbf{A} be a complete \mathbb{M}^ℓ -algebra. The following conditions are equivalent:*

- 1) \mathbf{A} satisfies the above infinitary law (2) of distributivity of $*$ over \vee ,
- 2) There is a (unique) operation \rightarrow defined on A satisfying the law of residuation.

Proof: One direction is proved taking the definition $a \rightarrow b = \bigvee \{c \in A : a * c \leq b\}$ for every $a, b \in A$. The other direction is a straightforward check. We stress that this proof is based on very few hypotheses on \mathbf{A} ; indeed the associativity of $*$ is not needed. \square

Proposition 3.8. *A complete \mathbb{M}^ℓ -algebra is the $\langle \vee, \wedge, *, \neg, 0, 1 \rangle$ -reduct of a residuated lattice if, and only if, it satisfies the infinitary distributive law (2).*

Proof: Let \mathbf{A} be a complete \mathbb{M}^ℓ -algebra. If \mathbf{A} is the reduct of a residuated lattice \mathbf{A}' , then \mathbf{A} satisfies (2) since \mathbf{A}' satisfies (2). Conversely, if \mathbf{A} satisfies (2) then the algebra $\langle \mathbf{A}, \rightarrow \rangle$, where \rightarrow is the operation given by Proposition 3.7, is easily checked to be a residuated lattice. \square

Corollary 3.9. *Every finite member of \mathbb{M}^{sl} or \mathbb{M}^ℓ is the reduct of a residuated lattice.*

Proof: All the algebras that the statement dealt with are complete and satisfy (2). This last part is proved by induction from Definition 3.1(3). Therefore, as a consequence of Propositions 3.6 and 3.8 we finish the proof. \square

Proposition 3.10. *There are complete \mathbb{M}^ℓ -algebras which are not the reduct of any residuated lattice.*⁸

Proof: We consider the residuated lattice $\mathbf{A} = \langle A, \vee, \wedge, *, \rightarrow, 0, 1 \rangle$ where $A = \{0, 1\} \cup \{x \in \mathbb{R} : \frac{1}{4} \leq x \leq \frac{3}{4}\}$, the lattice operations corresponds to the standard order over the real numbers and the other operations are defined by the following tables (where $a, b, c \in [\frac{1}{4}, \frac{3}{4}]_{\mathbb{R}}$ and $a < c$):

$*$	0	b	1
0	0	0	0
a	0	$\frac{1}{4}$	a
1	0	b	1

\rightarrow	0	a	c	1
0	1	1	1	1
a	0	1	1	1
c	0	$\frac{3}{4}$	1	1
1	0	a	c	1

We now we consider the algebra $\mathbf{B} = \langle B, \vee, \wedge, *, 0, 1 \rangle$ where $B = A \setminus \{\frac{3}{4}\}$ and the operations are the restrictions of the ones defined over A . It is clear that \mathbf{B} is a complete algebra and it is easy to check that it is a \mathbb{M}^ℓ -algebra. However, there is no possibility to define a residuation \rightarrow over B in such a way that its expansion becomes a residuated lattice. This is an immediate consequence of the fact that the generalized distributivity does not hold in \mathbf{B} , e.g.,

$$\left(\bigvee^{\mathbf{B}}[\frac{1}{4}, \frac{3}{4}]_{\mathbb{R}}\right) * \frac{1}{2} = 1 * \frac{1}{2} = \frac{1}{2} \text{ while } \bigvee^{\mathbf{B}}\{x * \frac{1}{2} : x \in [\frac{1}{4}, \frac{3}{4}]_{\mathbb{R}}\} = \bigvee^{\mathbf{B}}\{\frac{1}{4}\} = \frac{1}{4}. \quad \square$$

Thus we have already seen that there are \mathbb{M}^{sl} -algebras and \mathbb{M}^ℓ -algebras that are not the reduct of any residuated lattice. But, are they the subreduct of a certain residuated lattice? That is, are they, up to isomorphisms, equal to the class of the subalgebras of the reducts in the adequate languages of a residuated lattice? As a consequence of a result obtained by Ono (see [Ono03a] and references therein), we have that \mathbb{M}^{sl} and \mathbb{M}^ℓ are the classes of $\langle \vee, *, 0, 1 \rangle$ -subreducts and $\langle \vee, \wedge, *, 0, 1 \rangle$ -subreducts of the class \mathbb{RL} , respectively. Indeed, Ono shows (Cf. [Ono03a, Theorem 7]) that every \mathbb{M}^{sl} -algebra can be embedded in a complete residuated lattice in such a form that the embedding preserves all existing meets. So we have the following result.

Theorem 3.11. *\mathbb{M}^{sl} and \mathbb{M}^ℓ are the classes of $\langle \vee, *, 0, 1 \rangle$ -subreducts and $\langle \vee, \wedge, *, 0, 1 \rangle$ -subreducts of the class of residuated lattices, respectively.*

This last result will be used in Section 4 to show that $\mathcal{G}_{\text{ew}}[\vee, *]$ and $\mathcal{G}_{\text{ew}}[\vee, \wedge, *]$ are the $\langle \vee, *, 0, 1 \rangle$ -fragment and the $\langle \vee, \wedge, *, 0, 1 \rangle$ -fragment of \mathcal{G}_{ew} , respectively.

Let us recall that in [AV02, Theorem 7] it is shown that the variety \mathbb{M}^ℓ is not the equivalent algebraic semantics for any deductive system. It is easy to see that the same proof of [AV02] runs to show that neither the variety \mathbb{M}^{sl} is the equivalent algebraic semantics for any deductive system. We summarize these results in the next proposition.

Proposition 3.12. *The varieties \mathbb{M}^{sl} and \mathbb{M}^ℓ are not the equivalent algebraic semantics for any deductive system.*

Finally we will see that \mathbb{M}^{sl} and \mathbb{M}^ℓ have the finite embeddability property (FEP). This is an easy consequence from the fact that residuated lattices have the FEP (see [BvA02, Theorem 5.9]).

Let us recall that given an algebra $\mathbf{A} = \langle A, \langle f_i^A : i \in I \rangle \rangle$ of any type, and any non-empty subset $B \subseteq A$, the *partial subalgebra* \mathbf{B} of \mathbf{A} is the structure⁹

⁸The example used here is taken from [BGV05].

⁹Note that it is not an algebra since the operations may not be defined around all the universe. These structures have been sometimes called *partial algebras*.

$\langle B, \langle f_i^{\mathbf{B}} : i \in I \rangle \rangle$, where for every k -ary functional f_i , and $b_1, \dots, b_k \in B$,

$$f_i^{\mathbf{B}}(b_1, \dots, b_k) = \begin{cases} f_i^{\mathbf{A}}(b_1, \dots, b_k), & \text{if } f_i^{\mathbf{A}}(b_1, \dots, b_k) \in B, \\ \text{undefined}, & \text{if } f_i^{\mathbf{A}}(b_1, \dots, b_k) \notin B. \end{cases}$$

A class \mathbf{K} of algebras has the *finite embeddability property*, the FEP for short, if every finite partial subalgebra of each member of \mathbf{K} can be embedded in a finite member of \mathbf{K} .

Theorem 3.13. $\mathbb{M}^{s\ell}$ and \mathbb{M}^ℓ have the finite embeddability property. Therefore, its quasi-equational (and universal) theory is decidable.

Proof: It is enough to prove the first part. Let \mathbf{A} be any algebra in $\mathbb{M}^{s\ell}$ and let \mathbf{B} be a finite partial subalgebra of \mathbf{A} . By Theorem 3.11, \mathbf{A} is embeddable in a residuated lattice \mathbf{A}' . Let i be such an embedding. Now we have that $i[\mathbf{B}]$ is a finite partial subalgebra of \mathbf{A}' and thus, since \mathbb{RL} has the FEP, it can be embedded in a finite residuated lattice \mathbf{D} . Let h be this embedding and let \mathbf{D}' be the $\langle \vee, *, 0, 1 \rangle$ -reduct of \mathbf{D} . \mathbf{D}' is a finite $\mathbb{M}^{s\ell}$ -algebra and the map $h \circ i$ is an embedding from \mathbf{B} into \mathbf{D}' . A similar argument runs for \mathbb{M}^ℓ . \square

4 Connecting the logical systems and the algebras

This section studies the connections between the Gentzen systems $\mathcal{G}_{\mathbf{ew}}[\vee, *]$ and $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *]$, their external deductive systems $\mathcal{E}_{\mathbf{ew}}[\vee, *]$ and $\mathcal{E}_{\mathbf{ew}}[\vee, \wedge, *]$ and the varieties algebras $\mathbb{M}^{s\ell}$ and \mathbb{M}^ℓ . Let us start by recalling two results concerning $\mathcal{G}_{\mathbf{ew}}$, $IPC^* \setminus c$ and the variety \mathbb{RL} .

Theorem 4.1. (Cf. [AV00, Theorem 21]) \mathbb{RL} is the equivalent algebraic semantics of $IPC^* \setminus c$, with translations τ and ρ defined as follows: $\tau(p) = \{p \approx 1\}$ and $\rho(p \approx q) = \{p \rightarrow q, q \rightarrow p\}$.

Theorem 4.2. [AV00, Theorem 22] \mathbb{RL} is the equivalent algebraic semantics of $\mathcal{G}_{\mathbf{ew}}$, with translations τ and ρ defined as follows:

$$\tau(\varphi_0, \dots, \varphi_{m-1} \Rightarrow \varphi) := \begin{cases} \{\varphi_0 \rightarrow (\varphi_1 \rightarrow (\dots \rightarrow (\varphi_{m-1} \rightarrow \varphi) \dots)) \approx 1\}, & \text{if } m \geq 1 \\ \{\varphi \approx 1\}, & \text{if } m = 0 \end{cases}$$

$$\rho(\varphi \approx \psi) := \{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi\}.$$

Remark 4.3. Since residuated lattices satisfy that

- $(x_0 * x_1 * \dots * x_n) \rightarrow y \approx x_0 \rightarrow (x_1 \rightarrow (\dots \rightarrow (x_n \rightarrow y) \dots))$,
- $x \vee y \approx y$ iff $x \leq y$ iff $x \rightarrow y \approx 1$,

it holds that we can replace the translation τ in Theorem 4.2 in the case $m > 0$ with

$$\tau(\varphi_0, \dots, \varphi_{m-1} \Rightarrow \varphi) = \{(\varphi_0 * \dots * \varphi_{m-1}) \vee \varphi \approx \varphi\},$$

Notice that this new translation only uses the connectives $*$, \vee , 1 .

The algebraization of $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *]$ was proved in [AV02]¹⁰. Let us recall that result.

¹⁰In [AV02] the Gentzen system $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *]$ and the class \mathbb{M}^ℓ are denoted by $\mathcal{G}_{\wedge, \vee, *}$ and BCILM, respectively.

Theorem 4.4. (Cf. [AV02, Theorem 5]) *The Gentzen system $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *]$ is algebraizable, with equivalent algebraic semantics the variety \mathbb{M}^ℓ , with translations τ and ρ defined as follows:*

$$\tau(\varphi_0, \dots, \varphi_{m-1} \Rightarrow \varphi) = \begin{cases} \{(\varphi_0 * \dots * \varphi_{m-1}) \wedge \varphi \approx \varphi_0 * \dots * \varphi_{m-1}\}, & \text{si } m \geq 1 \\ \{1 \approx \varphi\}, & \text{si } m = 0 \end{cases}$$

$$\rho(\varphi \approx \psi) = \{\varphi \Rightarrow \psi; \psi \Rightarrow \varphi\}.$$

As \mathbb{M}^ℓ is a variety of lattices, it satisfies that the identity $x \wedge y \approx x$ is equivalent to $x \vee y \approx y$. Thus we can change the translation τ (for the case $m > 0$) in the following way:

$$\tau(\varphi_0, \dots, \varphi_{m-1} \Rightarrow \varphi) = \{(\varphi_0 * \dots * \varphi_{m-1}) \vee \varphi \approx \varphi\}.$$

So in this way we obtain a translation from $\langle \vee, *, 0, 1 \rangle$ -sequents to $\langle \vee, *, 0, 1 \rangle$ -equations. In [AV02] The system $\mathcal{G}_{\mathbf{ew}}[\vee, *]$ is not considered in [AV02], but the proof of [AV02, Theorem 11] also shows that $\mathcal{G}_{\mathbf{ew}}[\vee, *]$ is algebraizable with equivalent algebraic semantics the variety $\mathbb{M}^{s\ell}$. We summarize these comments in the following theorem.

Theorem 4.5. *The Gentzen systems $\mathcal{G}_{\mathbf{ew}}[\vee, *]$ and $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *]$ are algebraizable, with equivalent algebraic semantics the varieties $\mathbb{M}^{s\ell}$ and \mathbb{M}^ℓ , respectively, with translations τ and ρ defined as follows:*

$$\tau(\varphi_0, \dots, \varphi_{m-1} \Rightarrow \varphi) = \begin{cases} \{(\varphi_0 * \dots * \varphi_{m-1}) \vee \varphi \approx \varphi\}, & \text{if } m \geq 1 \\ \{1 \approx \varphi\}, & \text{if } m = 0 \end{cases}$$

$$\rho(\varphi \approx \psi) = \{\varphi \Rightarrow \psi; \psi \Rightarrow \varphi\}.$$

It is easy to check that if we add the contraction rule to the system $\mathcal{G}_{\mathbf{ew}}[\vee, *]$, then the new system $\mathcal{G}_{\mathbf{ewc}}[\vee, *]$ is algebraizable and the variety of bounded distributive lattices is its equivalent algebraic semantics (Cf. [RV93, Theorem 4.9]).

Now we will see that the Gentzen systems $\mathcal{G}_{\mathbf{ew}}[\vee, *]$ and $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *]$ are fragments of the Gentzen system $\mathcal{G}_{\mathbf{ew}}$.

Corollary 4.6. *$\mathcal{G}_{\mathbf{ew}}[\vee, *]$ is the $\langle \vee, *, 0, 1 \rangle$ -fragment of $\mathcal{G}_{\mathbf{ew}}$, and $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *]$ is the $\langle \vee, \wedge, *, 0, 1 \rangle$ -fragment of $\mathcal{G}_{\mathbf{ew}}$. That is,*

$$\mathcal{G}_{\mathbf{ew}}[\vee, *] = \langle \vee, *, 0, 1 \rangle\text{-}\mathcal{G}_{\mathbf{ew}} \quad \text{and} \quad \mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *] = \langle \vee, *, 0, 1 \rangle\text{-}\mathcal{G}_{\mathbf{ew}}.$$

Proof: Since the two cases are analogous we simply prove the first one. We have to prove that for every $\Phi \cup \{\varsigma\} \subseteq \text{Seq}_{\langle \vee, *, 0, 1 \rangle}^{\omega \times \{1\}}$,

$$\Phi \vdash_{\mathbf{FL}_{\mathbf{ew}}} \varsigma \quad \text{iff} \quad \Phi \vdash_{\mathbf{FL}_{\mathbf{ew}}[\vee, *]} \varsigma.$$

Let τ be the translation of $\mathcal{G}_{\mathbf{ew}}$ in $\models_{\mathbf{RL}}$ stated in Theorem 4.2, and let τ' be the translation of $\mathcal{G}_{\mathbf{ew}}[\vee, *]$ in $\models_{\mathbb{M}^{s\ell}}$ stated in Theorem 4.5. Then we have the following chain of equivalences:

$$\begin{aligned} \Phi \vdash_{\mathbf{FL}_{\mathbf{ew}}} \varsigma & \quad \text{iff} \\ \tau(\Phi) \models_{\mathbf{RL}} \tau(\varsigma) & \quad \text{iff} \\ \tau'(\Phi) \models_{\mathbf{RL}} \tau'(\varsigma) & \quad \text{iff} \\ \tau'(\Phi) \models_{\mathbb{M}^{s\ell}} \tau'(\varsigma) & \quad \text{iff} \\ \Phi \vdash_{\mathbf{FL}_{\mathbf{ew}}[\vee, *]} \varsigma. & \end{aligned}$$

The second equivalence is obtained by Remark 4.3, and the third one by Theorem 3.11. \square

The following corollary states that the Gentzen systems $\mathcal{G}_{\mathbf{ew}}[\vee, *]$ and $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *]$ are properly substructural.

Corollary 4.7. *The contraction rule is admissible neither in $\mathcal{G}_{\mathbf{ew}}[\vee, *]$ nor in $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *]$.¹¹*

Proof: We prove the theorem for $\mathcal{G}_{\mathbf{ew}}[\vee, *]$ (the other case is analogous). It is obvious that $\emptyset \vdash_{\mathcal{G}_{\mathbf{ew}}[\vee, *]} p, p \Rightarrow p * p$. We will see that $\emptyset \not\vdash_{\mathcal{G}_{\mathbf{ew}}[\vee, *]} p \Rightarrow p * p$ with the help of Theorem 4.5. Let $\mathbf{A} = \langle \{0, \frac{1}{2}, 1\}, \vee, *, 0, 1 \rangle$ the $\langle \vee, *, 0, 1 \rangle$ -reduct of the MV-algebra of three elements. This algebra obviously belongs to $\mathbb{M}^{s\ell}$ but $\frac{1}{2} \not\leq \frac{1}{2} * \frac{1}{2}$ (because $\frac{1}{2} * \frac{1}{2} = 0$). \square

As a consequence of this corollary, we obtain the following result

Corollary 4.8. *The Gentzen systems $\mathcal{G}_{\mathbf{ew}}[\vee, *]$ and $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *]$ are properly included in $\mathcal{G}_{\mathbf{ewc}}[\vee, *]$. That is, $\mathcal{G}_{\mathbf{ew}}[\vee, *] \leq \mathcal{G}_{\mathbf{ewc}}[\vee, *]$, and $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *] \leq \mathcal{G}_{\mathbf{ewc}}[\vee, *]$.*

The following corollary states that, in contrast to the case of $\mathcal{G}_{\mathbf{FLew}}$ (which is equivalent to $IPC^* \setminus c$), the Gentzen systems $\mathcal{G}_{\mathbf{ew}}[\vee, *]$ and $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *]$ are not equivalent to any deductive system.

Corollary 4.9. *The Gentzen systems $\mathcal{G}_{\mathbf{ew}}[\vee, *]$ and $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *]$ are not equivalent to any deductive system.*

Proof: By Theorem 4.5 and Proposition 3.12 \square

Corollary 4.10. *The Gentzen systems $\mathcal{G}_{\mathbf{ew}}[\vee, *]$ and $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *]$ are decidable, i.e., their sets of entailments of the form $\{\Gamma_i \Rightarrow \Delta_i : i \in I\} \vdash \Gamma \Rightarrow \Delta$, with I finite, are decidable.*

Proof: It is an immediate consequence of the algebraization and Theorem 3.13. \square

The following two theorems state that the external systems of $\mathcal{G}_{\mathbf{ew}}[\vee, *]$ and $\mathcal{G}_{\mathbf{ew}}[\vee, \wedge, *]$ are exactly the corresponding fragments of $IPC^* \setminus c$.

Theorem 4.11. *For all $\Sigma \cup \{\varphi\} \subseteq Fm_{\langle \vee, *, 0, 1 \rangle}$, it holds that*

$$\Sigma \vdash_{IPC^* \setminus c} \varphi \quad \text{iff} \quad \Sigma \vdash_{\mathcal{E}_{\mathbf{ew}}[\vee, *]} \varphi.$$

Proof: By using the fact that $IPC^* \setminus c$ is the external deductive system of $\mathcal{G}_{\mathbf{ew}}$ (Corollary 2.6) and Corollary 4.6 we have that

$$\Sigma \vdash_{IPC^* \setminus c} \varphi \quad \text{iff} \quad \{\emptyset \Rightarrow \psi : \psi \in \Sigma\} \vdash_{\mathbf{FLew}} \emptyset \Rightarrow \varphi \quad \text{iff} \quad \{\emptyset \Rightarrow \psi : \psi \in \Sigma\} \vdash_{\mathbf{FLew}[\vee, *]} \emptyset \Rightarrow \varphi.$$

And this is precisely what is claimed in the statement. \square

Theorem 4.12. *For all $\Sigma \cup \{\varphi\} \subseteq Fm_{\langle \vee, \wedge, *, 0, 1 \rangle}$, it holds that*

$$\Sigma \vdash_{IPC^* \setminus c} \varphi \quad \text{iff} \quad \Sigma \vdash_{\mathcal{E}_{\mathbf{ew}}[\vee, \wedge, *]} \varphi.$$

Proof: The proof is analogous to the previous one. \square

The algebraization results (Theorem 4.5) allow us to prove completeness theorems for $\mathcal{E}_{\mathbf{ew}}[\vee, *]$ and $\mathcal{E}_{\mathbf{ew}}[\vee, \wedge, *]$ with respect to the varieties $\mathbb{M}^{s\ell}$ and \mathbb{M}^ℓ respectively.

¹¹A rule r is *admissible* in a Gentzen system \mathcal{G} if for every $\langle \Phi, \varsigma \rangle \in r$ and every substitution e , the \mathcal{G} -derivability of all sequents in $\{e(\varsigma') : \varsigma' \in \Phi\}$ implies the \mathcal{G} -derivability of the sequent $e(\varsigma)$.

Theorem 4.13. *The variety \mathbb{M}^{sl} is an algebraic semantics for $\mathcal{E}_{ew}[\vee, *]$ with defining equation $p \approx 1$. And the variety \mathbb{M}^l is an algebraic semantics for $\mathcal{E}_{ew}[\vee, \wedge, *]$ with the same defining equation.*

Proof: We restrict ourselves to the first case (the second case is proved in [AV02, Theorem 10]). By Theorem 4.5 we know that for every $\Sigma \cup \{\varphi\} \subseteq Fm_{\langle \vee, *, 0, 1 \rangle}$,

$$\{\emptyset \Rightarrow \psi : \psi \in \Sigma\} \vdash_{\mathbf{FL}_{ew}[\vee, *]} \emptyset \Rightarrow \varphi \quad \text{iff} \quad \{1 \approx \psi : \psi \in \Sigma\} \vDash_{\mathbb{M}^{sl}} \varphi \approx 1.$$

By the definition of $\mathcal{E}_{ew}[\vee, *]$ we conclude that for every $\Sigma \cup \{\varphi\} \subseteq Fm_{\langle \vee, *, 0, 1 \rangle}$,

$$\Sigma \vdash_{\mathcal{E}_{ew}[\vee, *]} \varphi \quad \text{iff} \quad \{1 \approx \psi : \psi \in \Sigma\} \vDash_{\mathbb{M}^{sl}} \varphi \approx 1.$$

And this is precisely what is claimed in the statement. \square

5 Connection with the corresponding fragments of the classical logic

In this section we will show that the $\langle \vee, *, 0, 1 \rangle$, $\langle \vee, \wedge, *, 0, 1 \rangle$ and $\langle \vee, \wedge, 0, 1 \rangle$ -fragments of $IPC^* \setminus c$ are exactly the $\langle \vee, *, 0, 1 \rangle$, $\langle \vee, \wedge, *, 0, 1 \rangle$ and $\langle \vee, \wedge, 0, 1 \rangle$ -fragments of classical logic.

We will use the following axiomatization of the $\langle \vee, \wedge \rangle$ -fragment of classical logic.

Theorem 5.1. (Cf.[DP80, FV91]) *The $\langle \vee, \wedge \rangle$ -fragment of classical propositional logic is axiomatized by the following rules:*

- (R1) $\varphi \vee \varphi \vdash \varphi$
- (R2) $\varphi \vdash \varphi \vee \psi$
- (R3) $\varphi \vee \psi \vdash \psi \vee \varphi$
- (R4) $\varphi \vee (\psi \vee \gamma) \vdash (\varphi \vee \psi) \vee \gamma$
- (R5) $\varphi \wedge \psi \vdash \varphi$
- (R6) $\varphi \wedge \psi \vdash \psi \wedge \varphi$
- (R7) $\{\varphi, \psi\} \vdash \varphi \wedge \psi$
- (R8) $\varphi \vee (\psi \wedge \gamma) \vdash (\varphi \vee \psi) \wedge (\varphi \vee \gamma)$
- (R9) $(\varphi \vee \psi) \wedge (\varphi \vee \gamma) \vdash \varphi \vee (\psi \wedge \gamma)$

An easy adaptation of the proof of [DP80] allows us to conclude that the $\langle \vee, \wedge, 0, 1 \rangle$ -fragment of the classical propositional logic can be axiomatized by adding the axiom

$$(A1) \quad 1$$

and the rule

$$(R10) \quad 0 \vdash \varphi$$

to the previous axiomatization.

Let us denote by CPL^* the classical propositional logic in the language $\langle \vee, \wedge, *, \rightarrow, 0, 1 \rangle$, where the behaviour of $*$ is exactly the same as \wedge .

Theorem 5.2. *Let Ψ be any of the languages $\langle \vee, *, 0, 1 \rangle$, $\langle \vee, \wedge, *, 0, 1 \rangle$ or $\langle \vee, \wedge, 0, 1 \rangle$. Then, the Ψ -fragment of $IPC^* \setminus c$ is equal to the Ψ -fragment of CPL^* . That is,*

$$\Psi\text{-}IPC^* \setminus c = \Psi\text{-}CPL^*.$$

Proof: It is obvious that $\Psi\text{-IPC}^*\setminus c \leq \Psi\text{-CPL}^*$. To prove the theorem it is enough to check that each of the rules (R1)-(R9) in Theorem 5.1 ((A1) and (R10) are immediate) are derivable rules in each Ψ -fragment of $\text{IPC}^*\setminus c$.

Let $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_\Psi$. Let \vdash_Ψ be the consequence relation of $\Psi\text{-IPC}^*\setminus c$. By applying the definition of Ψ -fragment and Theorem 4.1 we have

$$\Gamma \vdash_\Psi \varphi \quad \text{iff} \quad \Gamma \vdash_{\text{IPC}^*\setminus c} \varphi \quad \text{iff} \quad \{\gamma \approx 1 : \gamma \in \Gamma\} \models_{\mathbb{RL}} \varphi \approx 1.$$

We will use this semantical characterization of the Ψ -fragments of $\text{IPC}^*\setminus c$ to check the derivability of the rules. In fact we will only check (R8) and (R9) (the other are obvious).

Let $\odot \in \{\wedge, *\}$. To check that (R8) and (R9) are derivable we have to show that

$$\varphi \vee (\psi \odot \gamma) \approx 1 \models_{\mathbb{RL}} (\varphi \vee \psi) \odot (\varphi \vee \gamma) \approx 1. \quad (3)$$

For this purpose we will apply the well known fact that every variety is generated as a quasivariety by its subdirectly irreducible members. So, a quasiequation holds in \mathbb{RL} if, and only if, it holds in every subdirectly irreducible residuated lattice. Thus, it will be sufficient to prove that the double inference (3) holds in this subclass of residuated lattices.

Let \mathbf{A} a subdirectly irreducible algebra of \mathbb{RL} . Let us recall that the subdirectly irreducible algebras of \mathbb{RL} have the following property (see [KO01, Proposition 1.4]):

$$\text{For all } a, b \in A, \text{ if } a \vee b = 1, \text{ then } a = 1 \text{ or } b = 1. \quad (4)$$

Now let $a, b, c \in A$ and suppose that $a \vee (b \odot c) = 1$. So, by (4), we have that $a = 1$ or $b \odot c = 1$. If $a = 1$, then $(a \vee b) \odot (a \vee c) = 1 \odot 1 = 1$ (since $1 \wedge 1 = 1$ and $1 * 1 = 1$). If $b \odot c = 1$ then $b = 1$ and $c = 1$ and so, we also have $(a \vee b) \odot (a \vee c) = 1 \odot 1 = 1$.

Conversely, suppose that $(a \vee b) \odot (a \vee c) = 1$. This is equivalent to having $a \vee b = 1$ and $a \vee c = 1$. By (4), $a \vee b = 1$ implies that $a = 1$ or $b = 1$ and $a \vee c = 1$ implies that $a = 1$ or $c = 1$. If $a = 1$ we have $a \vee (b \odot c) = 1$. If $a < 1$ we have $b = c = 1$ and so we also obtain $a \vee (b \odot c) = 1$.

□

Corollary 5.3. *Let Ψ be any of the languages $\langle \vee, *, 0, 1 \rangle$, $\langle \vee, \wedge, *, 0, 1 \rangle$ or $\langle \vee, \wedge, 0, 1 \rangle$. The Ψ -fragment of every t-norm based fuzzy logic is equal to the Ψ -fragment of classical logic.*

Proof: It is an immediate consequence of the fact that t-norm based fuzzy logics are axiomatic extensions of the monoidal logic $\text{IPC}^*\setminus c$ (see for instance [EGG03]). □

Notice that by Theorems 4.11 and 5.2 we have obtained the following

$$\mathcal{E}_{\text{ew}}[\vee, *] = \langle \vee, *, 0, 1 \rangle\text{-IPC}^*\setminus c = \langle \vee, *, 0, 1 \rangle\text{-CPC}^*.$$

These equalities allows us to obtain an alternative axiomatization of the $\langle \vee, *, 0, 1 \rangle$ -fragment of classical logic by means a sequent calculus *without contraction* as we summarize in the following result.

Corollary 5.4. *The $\langle \vee, \wedge, 0, 1 \rangle$ -fragment of classical logic is equal to the external deductive system associated to the Gentzen system defined by the following axioms and rules:*

Axioms:

$$\varphi \Rightarrow \varphi \quad (\text{Axiom 1}) \quad 0 \Rightarrow \varphi \quad (\text{Axiom 2}) \quad \emptyset \Rightarrow 1 \quad (\text{Axiom 3})$$

Structural rules:

$$\frac{\Gamma \Rightarrow \varphi \quad \varphi, \Pi \Rightarrow \xi}{\Gamma, \Pi \Rightarrow \xi} \quad (\text{Cut})$$

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \xi}{\Gamma, \psi, \varphi, \Pi \Rightarrow \xi} (e \Rightarrow) \quad \frac{\Gamma \Rightarrow \xi}{\varphi, \Gamma \Rightarrow \xi} (w \Rightarrow)$$

Rules of introduction of connectives:

$$\frac{\varphi, \Gamma \Rightarrow \xi \quad \psi, \Gamma \Rightarrow \xi}{\varphi \vee \psi, \Gamma \Rightarrow \xi} (\vee \Rightarrow) \quad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} (\Rightarrow \vee_1) \quad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi} (\Rightarrow \vee_2)$$

$$\frac{\varphi, \psi, \Gamma \Rightarrow \xi}{\varphi \wedge \psi, \Gamma \Rightarrow \xi} (\wedge \Rightarrow) \quad \frac{\Gamma \Rightarrow \varphi \quad \Pi \Rightarrow \psi}{\Gamma, \Pi \Rightarrow \varphi \wedge \psi} (\Rightarrow \wedge)$$

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References

- [Ad01] R.J. ADILLON. Contribució a l'estudi de les lògiques proposicionals intuicionistes i de Gödel sense contracció i dels seus fragments. *Ph. D. Thesis*. University of Barcelona, 2001.
- [AV00] R. ADILLON, V. VERDÚ. On a contraction-less intuitionistic propositional logic with conjunction and fusion. *Studia Logica, Special Issue on Abstract Algebraic Logic*, 65(1):11–30, 2000.
- [AV02] R. ADILLON, V. VERDÚ. On a substructural Gentzen system, its equivalent variety semantics and its external deductive system. *Bulletin of the Section of Logic*, 31(3):125–134, 2002.
- [Avr88] A. AVRON. The semantics and proof theory of linear logic. *Theoretical Computer Science*, 570:161–184, 1988.
- [BGV05] F. BOU, À. GARCÍA-CERDAÑA, V. VERDÚ. On two fragments with negation and without implication of the logic of residuated lattices. *Archive for Mathematical Logic*. To appear.
- [BGV05p] F. BOU, À. GARCÍA-CERDAÑA, V. VERDÚ. On two fragments with negation and without implication of the logic of residuated lattices. Mathematics Preprint Series 369, IMUB (University of Barcelona), February 2005. Available on line at <http://www.imub.ub.es/publications.html>.
- [BP89] W. J. BLOK, D. PIGOZZI. *Algebraizable logics*, volume 396 of *Mem. Amer. Math. Soc.* A.M.S., Providence, January 1989.
- [BvA02] W. J. BLOK, C. J. VAN ALTEN. The finite embeddability property for residuated lattices, pocrim and BCK-algebras. *Algebra Universalis*, 48(3):253–271, 2002.

- [CEG00] CIGNOLI, R., ESTEVA, F., GODO, L., AND TORRENS, A. Basic fuzzy logic is the logic of continuous t-norms and their residua. *Soft Computing* 4 (2000), 106–112.
- [DP80] K. DYRDA, T. PRUCNAL. On finitely based consequence determined by a distributive lattice. *Bulletin of the Section of Logic*, 9:60–66, 1980.
- [EG01] F. ESTEVA, L. GODO. Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124:271–288, 2001.
- [EGG03] F. ESTEVA, L. GODO, À. GARCÍA-CERDAÑA. On the hierarchy of t-norm based residuated fuzzy logics. In M. Fitting and Ewa Orłowska, editors, *Beyond two: theory and applications of multiple-valued logic*, volume 114 of *Studies in Fuzziness and Soft Computing*, pages 251–272. Physica, Heidelberg, 2003.
- [FV91] J. M. FONT, V. VERDÚ. Algebraic logic for classical conjunction and disjunction. *Studia Logica, Special Issue on Algebraic Logic*, 50:391–419, 1991.
- [GTV97] A. J. GIL, A. TORRENS, V. VERDÚ. On Gentzen systems associated with the finite linear MV-algebras. *Journal of Logic and Computation*, 7(4):473–500, 1997.
- [Go01] GOTTWALD, S. *A treatise on many-valued logics*, vol. 9 of *Studies in Logic and Computation*. Research Studies Press, Baldock, 2001.
- [GGCB03] GOTTWALD, S., GARCÍA-CERDAÑA, À., AND BOU, F. Axiomatizing monoidal logic. A correction to: “a treatise on many-valued logics”. *Journal of Multiple-Valued Logic and Soft Computing* 9, 4 (2003), 427–433.
- [Háj98] P. HÁJEK. *Metamathematics of fuzzy logic*, volume 4 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 1998.
- [Höh95] U. HÖHLE. Commutative, residuated l -monoids. In U. Höhle and E. P. Klement, editors, *Non-classical logics and their applications to fuzzy subsets (Linz, 1992)*, volume 32 of *Theory Decis. Lib. Ser. B Math. Statist. Methods*, pages 53–106. Kluwer Acad. Publ., Dordrecht, 1995.
- [JeMo02] JENEI, S., AND MONTAGNA, F. A proof of standard completeness for Esteva and Godo’s logic MTL. *Studia Logica* 70 (2002), 183–192.
- [JT02] P. JIPSEN, C. TSINAKIS. A survey of residuated lattices. In J. Martinez, editor, *Ordered Algebraic Structures*, pages 19–56. Kluwer Academic Publishers, Dordrecht, 2002.
- [KO00] T. KOWALSKI, H. ONO. The variety of residuated lattices is generated by its finite simple members. *Reports on Mathematical Logic*, (34): 59–77, 2000. Algebra & substructural logics (Tatsunokuchi, 1999).
- [KO01] T. KOWALSKI, H. ONO. Residuated lattices: an algebraic glimpse at logics without contraction (preliminary report). 2001.
- [OK85] H. ONO, Y. KOMORI. Logics without the contraction rule. *The Journal of Symbolic Logic*, 50(1): 169–201, 1985.

- [Ono98] H. ONO. Proof-theoretic methods for nonclassical logic - an introduction. In M. Takahashi, M. Okada, and M. Dezani-Ciancaglini, editors, *Theories of Types and Proofs*, MSJ Memoirs 2, pages 207–254. Mathematical Society of Japan, 1998.
- [Ono03a] H. ONO. Closure operators and complete embeddings of residuated lattices. *Studia Logica*, 74(3): 427–440, 2003.
- [Ono03b] H. ONO. Completions of algebras and completeness of modal and substructural logics. volume 4 of *Advances in Modal Logic*, pages 335–353. 2003.
- [Ono03c] H. ONO. Substructural logics and residuated lattices - an introduction. In V. F. Hendricks and J. Malinowski, editors, *50 Years of Studia Logica*, volume 21 of *Trends in Logic—Studia Logica Library*, pages 193–228. Dordrecht, 2003.
- [RV93] J. REBAGLIATO, V. VERDÚ. On the algebraization of some Gentzen systems. *Fundamenta Informaticae, Special Issue on Algebraic Logic and its Applications*, 18:319–338, 1993.
- [RV95] J. REBAGLIATO, V. VERDÚ. Algebraizable Gentzen systems and the deduction theorem for Gentzen systems. Mathematics Preprint Series 175, University of Barcelona, June 1995.

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