Hardy-Sobolev Spaces and Algebras of Holomorphic Functions on the Unit Ball in $\mathbb{C}^n$

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Abstract

In a 1994 paper, U. Klein proved that for $1 \le p \le \infty$ and $m = 1, 2, \ldots$, the spaces H^p_m consisting of all holomorphic functions f on the unit disc for which $f^{(m)}$ is in the classical Hardy space H^p are Banach algebras under suitable norms.

In the talk we will consider extensions of this result to Hardy-Sobolev spaces on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$, including the case $0 < p \le 1$. For $m = 1, 2, \ldots$, $0 < p \le \infty$, the Hardy-Sobolev space $H^p_m(\mathbb{B}_n)$ is the space of holomorphic functions f on $\mathbb{B}_n$ for which the m^{th} order radial derivative $\mathcal{R}^m f$ is in the Hardy space $H^p(\mathbb{B}_n)$. The main result is as follows. **Theorem.** *Let* $m \in \{1, 2, 3, \ldots\}$. *If*

(a) $m \ge n$ *and* $\frac{n}{m} \le p \le \infty$, *or*

(b) $1 \le m < n$ $(n \ge 2)$ *and* $\frac{n}{m} < p \le \infty$,

then $H^p_m(\mathbb{B}_n)$ *is an algebra. Furthermore,* $\| \, \|_{p,m,\lambda}$ *defined for* $p \ge 1$ *by*

$$\|f\|_{p,m,\lambda} = \|f\|_\infty + \sum_{k=1}^{m} \lambda_k \|\mathcal{R}^k f\|_{\frac{m}{k}p}, \quad f \in H^p_m(\mathbb{B}_n),$$

where $\{\lambda_k\}_{k=1}^m$ *is a sequence of positive numbers satisfying*

$$\lambda_k \begin{pmatrix} k \\ j \end{pmatrix} \le \lambda_j \lambda_{k-j}, \quad j = 1, \ldots, k,$$

is a Banach algebra norm on $H^p_m(\mathbb{B}_n)$ *whenever* p, m *with* $p \ge 1$ *satisfy one of (a) or (b) above.*

We also prove that if $m > n$ and $\frac{n}{m} \le p < 1$, then $H^p_m(\mathbb{B}_n)$ is a p-Banach algebra under a suitable p-norm on $H^p_m(\mathbb{B}_n)$. By examples it is shown that the results are best possible. The talk will also consider extensions of these results to weighted Bergman Sobolev spaces.

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