Ugo LOCATELLI

ON THE EFFECTIVE STABILITY IN THE NEIGHBOURHOOD OF KAM TORI

Work in collaboration with

Alessandra Celletti
Dipartimento di Matematica,
Università di Roma ‘‘Tor Vergata’’

Antonio Giorgilli
Dipartimento di Matematica,
Università degli studi di Milano

Introduction

• The set of KAM tori does not contain any open set. Therefore, until 15 years ago, KAM theorem was thought to be able to ensure the stability just for systems with **2 degrees of freedom** (=DOF), thanks to a topological confinement.

• For Hamiltonian systems with **more than 2 DOF**, Nekhoroshev's theorem was supposed to be the best tool to prove the *"effective" stability*. In fact, it is able to provide upper bounds to the eventual diffusion of the actions variables **for very long times**.

• In Morbidelli A. & Giorgilli A.: "Superexponential stability of KAM tori", *J. Stat. Phys.* (1995), KAM and Nekhoroshev's theorems are combined so that **the invariant tori are shown to be in an excellent position for proving the** *"effective" stability* **nearby** (in problems with more than 2 DOF).

• Here, we want to reconsider the approach due to Morbidelli & Giorgilli, in order to **evaluate its applicability to concrete physical systems**.

What has been done in the past *(theory)*

• Proof scheme due to Morbidelli & Giorgilli. Start from a quasi-integrable Hamiltonian

$$H(\underline{p},\underline{q}) = h(\underline{p}) + \varepsilon f(\underline{p},\underline{q}) \ ,$$

where $(\underline{p},\underline{q}) \in \mathbf{R}^n \times \mathbf{T}^n$ and ε is a small parameter.
(1) Construct the Kolmogorov's normal form:

$$H(\underline{p},\underline{q}) = \underline{\omega} \cdot \underline{p} + \mathcal{O}(\|\underline{p}\|^2) \ ,$$

where $\underline{\omega}$ is a fixed, Diophantine frequency vector, i.e. $|\underline{k} \cdot \underline{\omega}| \geq \gamma / |\underline{k}|^\tau \ \forall \ \underline{k} \in \mathbf{Z}^n \setminus \{\underline{0}\}$.
(2) Consider the distance from the invariant torus $\rho = \|\underline{p}\|$ as a new "small parameter" and construct the Birkhoff's normal form up to an "optimal order":

$$H(\underline{p},\underline{q}) = \underline{\omega} \cdot \underline{p} + \sum_{l=1}^{r_{opt}} Z_l(\underline{p}) + \mathcal{R}(\underline{p},\underline{q}) \ ,$$

with $\mathcal{R}(\underline{p},\underline{q}) = \mathcal{O}(\|\underline{p}\|^{r_{opt}+2})$ and r_{opt} such that

$$\sup_{(\underline{p},\underline{q}) \in B_\rho(0) \times \mathbf{T}^n} \left| \mathcal{R}(\underline{p},\underline{q}) \right| \lesssim \exp\left(- \left(\frac{\rho_*}{\rho} \right)^{1/(\tau+1)} \right) ,$$

where ρ_* is a positive constant.

What has been done in the past *(theory)*

(3a) Consider the complementary set $\mathcal{T}^c(\rho)$ of the invariant tori belonging to $B_\rho(0)$. If the quadratic part $Z_1(\underline{p})$ of the normalized Hamiltonian is non-degenerate, then $\mathrm{Vol}\big(\mathcal{T}^c(\rho)\big) \propto \sqrt{\|\mathcal{R}\|}$ (see Neishtadt A., PMM U.S.S.R. (1982)) and

$$\mathrm{Vol}\big(\mathcal{T}^c(\rho)\big) \lesssim \exp\left(-\frac{\frac{1}{2}\,(\rho_*)^{1/(\tau+1)}}{\rho^{1/(\tau+1)}}\right) .$$

(3b) Assume that the Hamiltonian in Birkhoff's normal form is also quasi-convex. Therefore, we can apply the Nekhoroshev's theorem in the version provided by Pöschel J. (Math. Zeitsc., 1993). Thus, if the initial condition $\underline{p}_0 \in B_\rho(0)$, then $\|\underline{p}(t) - \underline{p}_0\|$ will remain "exponentially small" for all

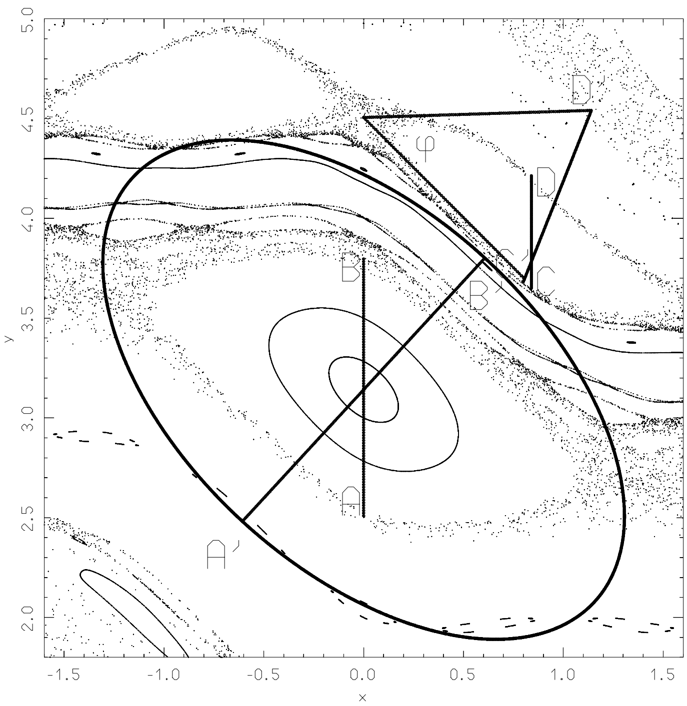
$$|t| \le T_d \sim \exp\left[C \exp\left(\frac{\frac{1}{2n}\,(\rho_*)^{1/(\tau+1)}}{\rho^{1/(\tau+1)}}\right)\right] ,$$

where C is a positive constant. Let us stress that the "diffusion time" T_d is proportional to the *exponential of the exponential of the inverse of the distance* ρ *from the KAM torus* related to the frequency vector $\underline{\omega}$.

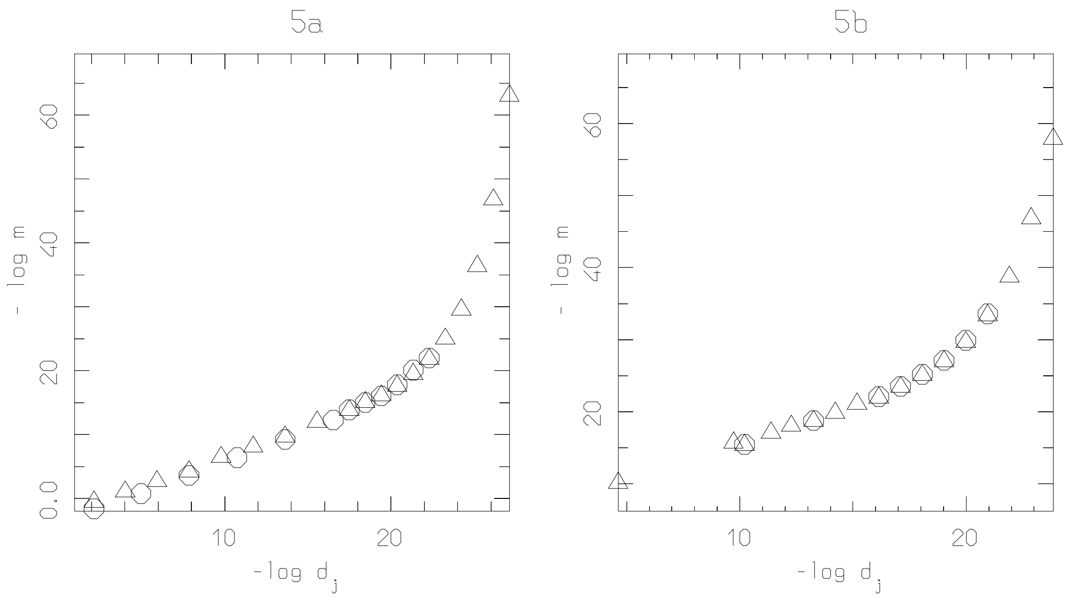
What has been done in the past
(numerical experiments on mappings)

By numerically explorating the standard map *close enough to the golden torus*, Lega E. & Froeschlé C. (*Physica D*, 1996) showed that the size of the resonant regions shrinks exponentially to zero with respect to the distance of the golden torus itself.

In L.U., Lega E., Froeschlé C. & Morbidelli A., *Physica D*, **139** (2000), the Greene's method is adapted so to approximate the size of the resonant islands via the computation of the residue.



What has been done in the past
(numerical experiments on mappings)



For each of the figures above a frequency ω is fixed. The size m_j of the resonance related to the j–th best approximant P_j/Q_j of ω is studied as a function of the distance $d_j = |\omega - P_j/Q_j|$. The approximations provided by the calculation of the residue (symb. $\triangle$) *nicely agree* with the results given by a frequency analysis method (symb. $\bigcirc$). Moreover, from the Greene's conjecture, one can guess the law:

$$m_j \simeq c'_1 d_j \exp\left(-c'_2 \big/ \sqrt{d_j}\right) ,$$

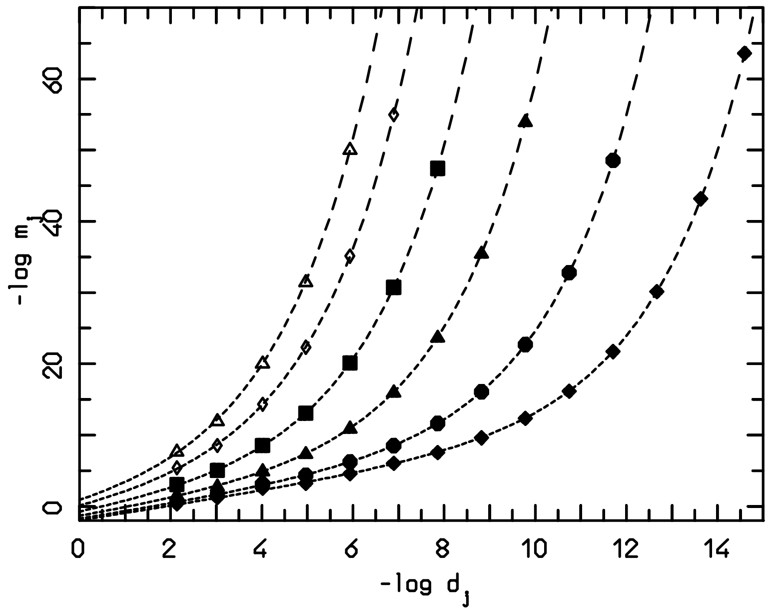
with c'_1 , c'_2 suitable positive constants.

NEW NUMERICAL EXPERIMENTS

Focus on a model of a forced pendulum, i.e.

$$H_{2D}(p,q,t) = \frac{1}{2}p^2 + \varepsilon\,[\cos q + \cos(q-t)] \ .$$

By iterating $2\pi/h$ times the leap-frog integrator (with time-step h) of the flow induced by H_{2D}, we can introduce a Poincaré map $M_\varepsilon : \mathbf{R}\times\mathbf{T} \mapsto \mathbf{R}\times\mathbf{T}$ that is symplectic. Thus, we can repeat the numerical experiments previously described.



In fig. above, each symbol corresponds to a value of ε. The dashed curves are drawn according to the asymptotic law $m_j \simeq c_1' d_j \exp\left(-c_2' \big/ \sqrt{d_j}\right)$, with c_1', c_2' given by a least squares fit.

Remark: the parameter c_2', ruling the exponential decrease of the resonant regions, *can be measured* with such a *numerical experiment*.

Remark: the *analytical theory* can evaluate another parameter ρ_*, ruling the exponential decrease of the resonant regions. Moreover, the superexponential estimate about the "diffusion time" depends on that same parameter.

QUESTION: *how far* are the *analytical estimates* from the *numerical measures* about the exponential decrease of the resonant regions?

Remark: computer assisted proofs can be successfully implemented in order to perform the initial construction of the Kolmogorov's normal form for realistic values of ε.

Remark: in order to produce *explicit* analytical estimates that can suitably apply in a *computer-assisted context*, we are forced to *partially rewrite them*. Basically, this requires to *adapt the standard technique producing the estimates for the Birkhoff's normal form*.

BIRKHOFF'S NORMAL FORM
(constructive algorithm)

• Start with a Hamiltonian of the following type:

$$\begin{aligned}\mathcal{H}^{(r-1)}(\underline{p},\underline{q}) &= \underline{\omega}\cdot\underline{p} + Z_1(\underline{p}) + \ldots + Z_{r-1}(\underline{p}) \\ &\quad + \sum_{l=r}^{\infty} f_l^{(r-1)}(\underline{p},\underline{q}) \ ,\end{aligned}$$

where $Z_l(\underline{p})$ and $f_l^{(r-1)}(\underline{p},\underline{q})$ are homogeneous polynomials of degree $l+1$ with respect to $\underline{p}$.

• Determine a generating function $\chi_r(\underline{p},\underline{q})$ by solving the homological equation

$$\sum_{j=1}^{n} \omega_j \frac{\partial \chi_r}{\partial q_j} + f_r^{(r-1)}(\underline{p},\underline{q}) = Z_r(\underline{p}) \ .$$

• The next Hamiltonian is defined as

$$\mathcal{H}^{(r)} = \exp \mathcal{L}_{\chi_r} \mathcal{H}^{(r-1)} \ ,$$

being $\exp \mathcal{L}_{\chi_r} \cdot$ the usual Lie series operator.

• By gathering all the summands having the same degree in $\underline{p}$, one obtains iterative formulas to calculate the new terms entering the expansion

$$\begin{aligned}\mathcal{H}^{(r)}(\underline{p},\underline{q}) &= \underline{\omega}\cdot\underline{p} + Z_1(\underline{p}) + \ldots + Z_{r-1}(\underline{p}) + Z_r(\underline{p}) \\ &\quad + \sum_{l=r+1}^{\infty} f_l^{(r)}(\underline{p},\underline{q}) \ .\end{aligned}$$

BIRKHOFF'S NORMAL FORM

(scheme of estimates)

• When the homological equation is solved, the Diophantine inequality implies that

$$\|\chi_r\| \propto r^{\tau} \left\| f_r^{(r-1)} \right\| .$$

• *Roughly speaking*, the derivatives due to the Poisson brackets add some factors $\mathcal{O}(r)$, then

$$\left\| f_{r+1}^{(r)} \right\| \propto \|\mathcal{L}_{\chi_r} Z_1\| \propto r\, \|\chi_r\| \lesssim r^{\tau+1} \left\| f_r^{(r-1)} \right\| .$$

Iterating such estimates, $f_{r+1}^{(r)} = \mathcal{O}\Big((r!)^{\tau+1}\Big)$.

Remark: this scheme of estimates is easy to prove for nonlinear oscillators, but it needs some additional (*standard*) analytic work near a torus.

• The accumulation of the factors $\mathcal{O}(r)$ is so that the following estimate hold when $\underline{p} \in B_{\rho}(0)$:

$$\left\| \mathcal{R}^{(r)} \right\| = \left\| \sum_{l=r+1}^{\infty} f_l^{(r)} \right\| \lesssim (r!)^{\tau+1} \rho^r .$$

• If $r = r_{opt} = r_{opt}(\rho)$ minimizing $(r!)^{\tau+1}\rho^r$, then

$$\left\| \mathcal{R}^{(r_{opt})} \right\| \lesssim \exp\left(- \left(\frac{\rho_*}{\rho} \right)^{1/(\tau+1)} \right) .$$

BIRKHOFF'S NORMAL FORM
(final estimates near a KAM torus)

By applying this technique, we can prove that

$$\sup_{(\underline{p},\underline{q})\in B_\rho(0)\times\mathbf{T}^n} \left|\mathcal{R}^{(r_{opt})}(\underline{p},\underline{q})\right| \le C\rho^2 \exp\left(-\left(\frac{\rho_*}{\rho}\right)^{\frac{1}{\tau+1}}\right) ,$$

where C is a constant and

$$\rho_* = \frac{\frac{\gamma}{\mathcal{M}}\left(\frac{\bar{d}}{2}\right)^{\tau+2}\sigma^{\tau+1}}{\left[2^{\tau+2}e^2(R+1)\right]^{1/(R+1)}\left(\Theta+4\right)} ,$$

with σ equal to the width of the analytic strip in the angles, $\bar{d} = \ldots$, $R = \ldots$ and so on. Briefly, ρ_* *can be explicitly calculated*.

Remark: our statement provides also suitable estimates about the normal form terms $Z_s(\underline{p})$ with $s \ge 2$ (i.e. terms of higher degree than the quadratic ones). These inequalities are essential in order to eventually extend both the *non-degeneracy* and *convexity* properties from the quadratic part to the whole normal form. This is essential to apply the statements given by Neishtadt and Pöschel, respectively.

THE COMPLEMENTARY SET OF KAM TORI

(procedure comparing analytical results to numerics)

- As a *first application to a 2 DOF problem*, we started from the Kolmogorov's normal form related to the forced pendulum, i.e.

$$\mathcal{H}^{(0)}(\underline{p},\underline{q}) = p_0 + \omega\, p_1 + f_1^{(0)}(p_1, q_0, q_1\, ; \varepsilon)\ ,$$

where $\omega = (\sqrt{5}-1)/2 \Rightarrow \tau = 1$ and the computer-assisted estimate of the norm of the (quadratic) term $f_1^{(0)}$ is taken from Celletti A., Giorgilli A. & L.U., *Nonlinearity* (2000).

Remark: the *analytic* asympotic law about the volume of the complementary set of the KAM tori is *analogous* to that *guessed* by the Greene conjecture.

- We considered a few values of $\varepsilon < 0.0276$ (i.e., *less than the breakdown threshold*). For each of them, the value of the coefficient ruling the exponential decay of the resonant regions as given by the *analytic* theory is compared to that given by the *numerics*.

THE COMPLEMENTARY SET OF KAM TORI

(results comparing analytics to numerics)

The *analytical results* are reported in the second column of the table below. The ratios *comparing the analytical results to the numerical ones* are reported in the fourth column.

ε	ρ_*	c_2'	$\frac{(1-2\bar{d})\rho_*}{8\,{c_2'}^2}$
0.000025	3.09×10^{-5}	2.34	2.5×10^{-7}
0.00025	1.11×10^{-5}	1.58	2.0×10^{-7}
0.0025	1.79×10^{-6}	0.809	1.2×10^{-7}
0.01	1.41×10^{-7}	0.345	5.3×10^{-8}
0.02	2.71×10^{-9}	0.111	9.8×10^{-9}
0.025	1.40×10^{-12}	0.0345	5.2×10^{-11}

Remark: our approach *(in the present form)* might be applied to quasi-integrable systems subject to *extremely small* perturbations.

QUESTION: when it is interesting to apply these estimates to physical systems?

Answer: the neighborhood of a fixed KAM torus where the estimates holds true should include a set of initial conditions taking into account their uncertainities.

As an example, a *rough and large evaluation* of the uncertainities on the observational data about the *planetary motions* claims that *the initial conditions should be contained in a ball having a radius large* 10^{-6} *in actions* (see Giorgilli A., L.U. & Sansottera M.: "Kolmogorov and Nekhoroshev theory for the problem of three bodies", *submitted*).

Remark: therefore, the comparisons with the numerical experiments look *not so ridicolous*, but the forced pendulum problem is just *a toy model*.

THE COUPLED FORCED PENDULUMS

(beginning of numerical experiments)

• Consider two coupled forced pendulums (i.e. a *3 DOF problem*):

$$H_{4D}(\underline{p},\underline{q},t) \;=\; \frac{1}{2}\left(p_1^2+p_2^2\right)+\varepsilon\Big[\cos(q_1-t) + \cos(q_2-t)+b\cos(q_1-q_2)\Big]\,.$$

• We focus on a neighborhood of the KAM torus characterized by $\underline{\omega} = (1, 1/\alpha, \alpha)$, where $\alpha \simeq 1.3247$ is the unique real solution of the equation $x^3 - x - 1 = 0$, then $\underline{\omega}$ is Diophantine with $\tau = 2$. We limit ourselves to consider the coupling value $b = 0.4$.

• The adaptation of the Greene method to symplectic maps in more than 2D (see Tompaidis S., *Exp. Math.* (1996) or Celletti A., Falcolini C. & L.U., *Reg. Chaot. Dyn.* (2004)) can be applied to the Poincaré map of the flow induced by H_{4D}. It provides $\varepsilon = 0.045 \pm 0.005$ as the breakdown threshold for the KAM torus corresponding to $\underline{\omega}$. This evaluation is confirmed by the frequency analisys method as shown in the following figures.

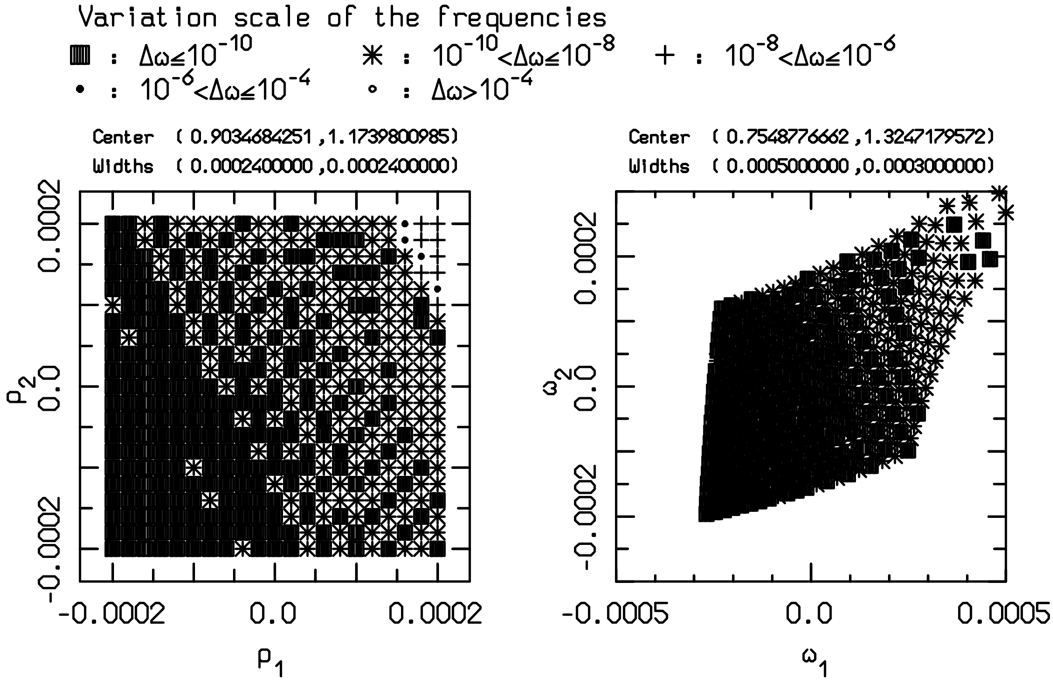
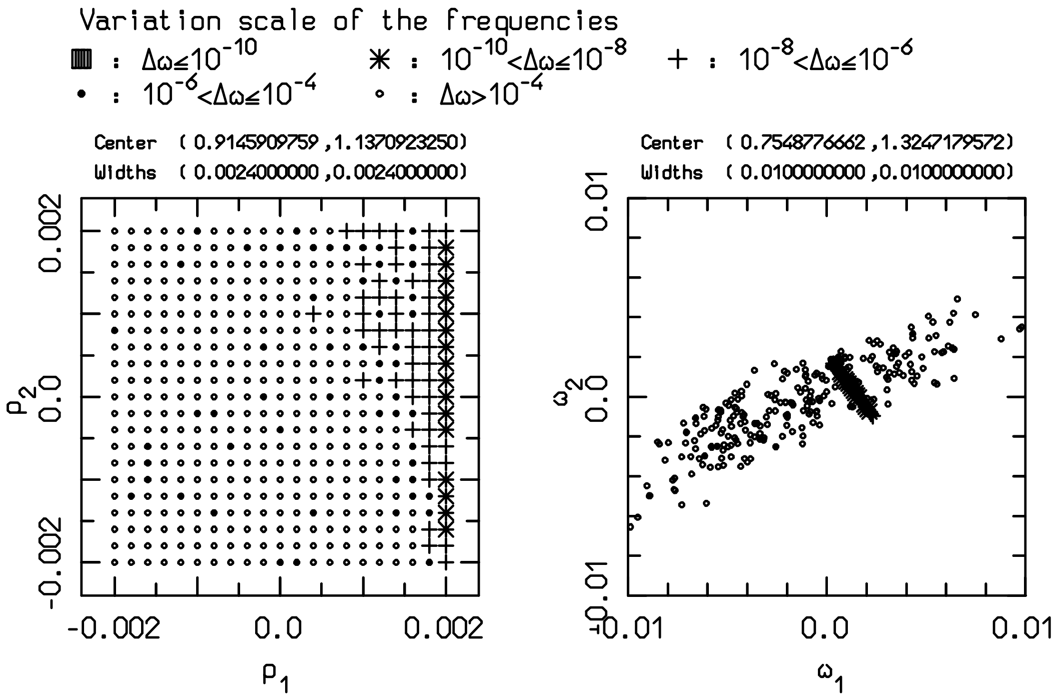


Figure above shows that the KAM torus related to $\underline{\omega}$ *exists* when $\varepsilon = 0.04$, while it *does not exist* when $\varepsilon = 0.05$, as shown in figure below.



TOWARDS NEKHOROSHEV'S THEOREM

(checking the hypotheses)

• For some fixed value of ε , we *prove (in a computer assisted way)* that the Hamiltonian can be lead in the following Kolmogorov's normal form (even with $\varepsilon > 0.02$):

$$\begin{aligned}\mathcal{H}^{(0)}(\underline{p},\underline{q}) \;=\; & p_0 + \omega_1 p_1 + \omega_2 p_2 \\ & + f_1^{(0)}(p_1, p_2, q_0, q_1, q_2) \ .\end{aligned}$$

• Let matrix A be such that $\frac{1}{2}A\underline{p}\cdot\underline{p} = \langle f_1^{(0)} \rangle$. The quasi-convexity property requires that

$$|\underline{\omega}\cdot\underline{v}| > \lambda\|\underline{v}\| \quad \text{or} \quad A\underline{v}\cdot\underline{v} \ge \mu\|\underline{v}\|^2$$

for some fixed $\lambda > 0$, $\mu > 0$ and $\forall\,\underline{v}$.

• Our statement about the Birkhoff's normal form allows us to extend the quasi-convexity property to all the normalized part, then we can apply the Nekhoroshev's theorem. This requires that the action radius ρ is small enough; the most demanding restriction is of the type

$$\rho < \rho_* \big/ (-\log\zeta)^{\tau+1} \ ,$$

where ζ is an extremely small quantity.

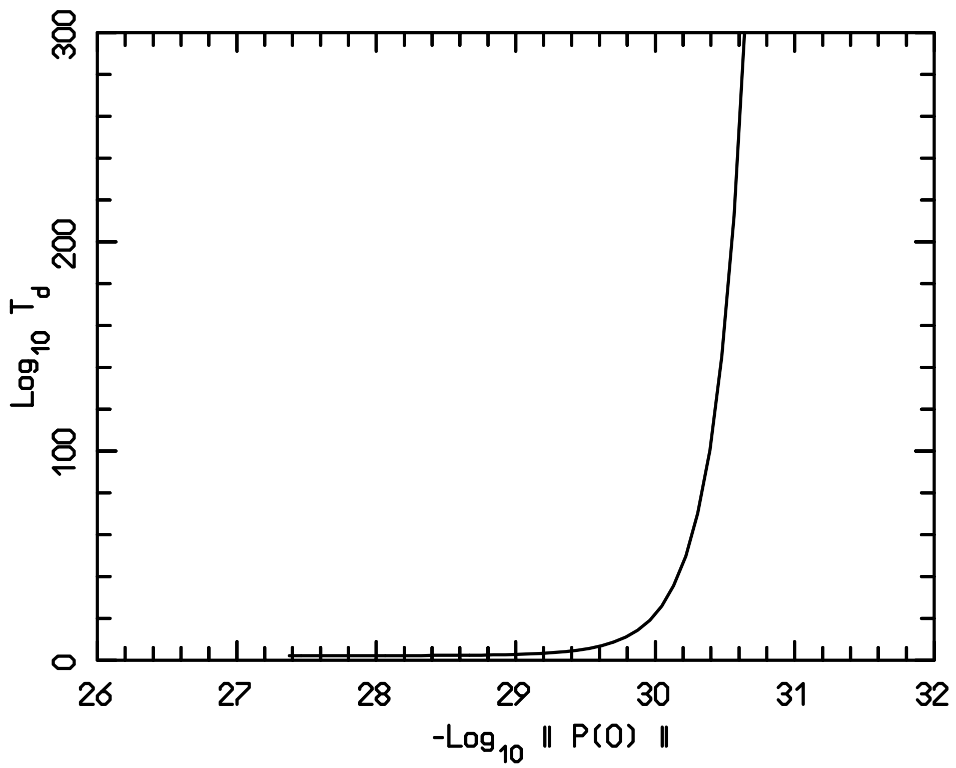
DIFFUSION TIMES: LOWER BOUNDS

• Finally, we can ensure that the drift in actions is *exponentially small* for all times $|t| \leq T_d$, with

$$T_d = C_1 \exp\left\{ C_2 \exp\left[\frac{1}{2n} \left(\frac{\rho_*}{\rho} \right)^{1/(\tau+1)} \right] \right\} ,$$

where C_1, C_2 and ρ_* *are explicitly calculated*.

• Consider all the KAM tori related to Diophantine frequency vectors $\underline{\omega}$ with $\tau = 8$ and in a ball of radius 10^{-10} centered about $(1, 1/\alpha, \alpha)$. The behaviour of $T_d(\rho)$ is reported in figure below in the case $\varepsilon = 0.00004$.



FINAL RESULT: the coupled forced pendulums with $\varepsilon = 0.00004$ is an *"effectively stable" system* when the initial condition stay in a suitable ball having a radius in actions of about 10^{-10}.

CONCLUSIONS: PROS & CONS

• The approach leading to the superexponential estimates **can produce explicit lower bounds to the diffusion times**.

• The constraints about the smallness of the actions radius are so restrictive that the final estimates **cannot (yet) apply to realistic physical systems**.

• Our comparisons clearly points the estimates in the **Birkhoff's normal form** (i.e. the evaluation of ρ_*) as the **main source of limitations**.